Eigenvalues and Eigenvectors

Problem Sheet, Friday 18.08.2023 (Room U358)

Question 1: Find the eigenvalues of a 2×2 matrix with trace 0 and determinant 1, using the facts that the trace of a matrix is equal to the sum of its eigenvalues, and the determinant is the product of its eigenvalues.

Answer: Let the eigenvalues be λ_1 and λ_2.

Given:

1. Trace = $\lambda_1 + \lambda_2 = 0$ (from the fact about the trace). From this, $\lambda_2 = -\lambda_1$.
2. Determinant = $\lambda_1 \times \lambda_2 = 1$ (from the fact about the determinant).

Using the relation from (1) in (2):

$$\lambda_1 \times (-\lambda_1) = 1 \implies -\lambda_1^2 = 1 \implies \lambda_1^2 = -1$$

From this, we get:

$$\lambda_1 = i, \quad \lambda_2 = -i$$

Thus, the eigenvalues of the matrix are i and $-i$.

Question 2: If λ is an eigenvalue of the matrix A, prove that λ^n is an eigenvalue of A^n for $n \geq 2$.

Answer:

Base case (n=2): Given A has an eigenvalue λ with corresponding eigenvector v, we have:

$$Av = \lambda v$$

Multiplying both sides by A:

$$A^2 v = \lambda A v = \lambda^2 v$$

Thus, if λ is an eigenvalue of A, λ^2 is an eigenvalue of A^2. This establishes our base case.

Inductive Step: Assume that the statement holds for some integer $k \geq 2$, i.e., $A^k v = \lambda^k v$. We need to prove that it holds for $k+1$.

Considering $A^{k+1} v$:

$$\begin{aligned} A^{k+1} v &= A(A^k v) \\ &= A(\lambda^k v) \\ &= \lambda^k (Av) \\ &= \lambda^k (\lambda v) \\ &= \lambda^{k+1} v \end{aligned}$$

This proves that if the statement is true for $n = k$, it's also true for $n = k+1$.

By the principle of mathematical induction, if λ is an eigenvalue of the matrix A, then λ^n is an eigenvalue of A^n for all integers $n \geq 2$.

Question 3: Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

Answer: To find its eigenvalues, compute the determinant of $A - \lambda I$:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 3 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix}$$

Expanding the determinant:

$$\det(A - \lambda I) = (1-\lambda)^3 + 3(1-\lambda) = (1-\lambda)((1-\lambda)^2 + 3)$$

This polynomial has the roots $\lambda_1 = 1$, $\lambda_2 = 1 - \sqrt{3}\,i$ and $\lambda_3 = 1 + \sqrt{3}\,i$, which are the eigenvalues of matrix A.

Eigenvalue $\lambda_1 = 1$: To determine the associated eigenvectors, we compute:

$$(A - I)v = 0,$$

where $A - I = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$. Solving the linear system

$$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

we find that $x = 0$, $z = 0$ and y being arbitrary. Thus, the eigenspace for λ_1 is given by

$$N(A - I) = \left\{ \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} \mid y \in \mathbb{R} \right\} = \left\{ y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \middle| \, y \in \mathbb{R} \right\},$$

yielding the representative eigenvector $v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Eigenvalue $\lambda_2 = 1 - \sqrt{3}\,i$: Similarly, we compute:

$$(A - (1 - \sqrt{3}\,i)I)v = 0,$$

where $A - (1 - \sqrt{3}\,i)I = \begin{bmatrix} \sqrt{3}\,i & 0 & 3 \\ 0 & \sqrt{3}\,i & 0 \\ -1 & 0 & \sqrt{3}\,i \end{bmatrix}$. For the system

$$\begin{bmatrix} \sqrt{3}\,i & 0 & 3 \\ 0 & \sqrt{3}\,i & 0 \\ -1 & 0 & \sqrt{3}\,i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

we deduce $x = \sqrt{3}\,iz$ and $y = 0$. The eigenspace for λ_2 is then

$$N(A - (1 - \sqrt{3}\,i)I) = \left\{ \begin{bmatrix} \sqrt{3}\,iz \\ 0 \\ z \end{bmatrix} \mid z \in \mathbb{R} \right\} = \left\{ z \begin{bmatrix} \sqrt{3}\,i \\ 0 \\ 1 \end{bmatrix} \middle| \, z \in \mathbb{R} \right\},$$

yielding the representative eigenvector $v_2 = \begin{bmatrix} \sqrt{3}\,i \\ 0 \\ 1 \end{bmatrix}$.

Eigenvalue $\lambda_3 = 1 + \sqrt{3}\,i$: Similarly, we compute:

$$(A - (1 + \sqrt{3}\,i)I)v = 0,$$

where $A - (1 + \sqrt{3}\,i)I = \begin{bmatrix} -\sqrt{3}\,i & 0 & 3 \\ 0 & -\sqrt{3}\,i & 0 \\ -1 & 0 & -\sqrt{3}\,i \end{bmatrix}$. For the system

$$\begin{bmatrix} -\sqrt{3}\,i & 0 & 3 \\ 0 & -\sqrt{3}\,i & 0 \\ -1 & 0 & -\sqrt{3}\,i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

we deduce $x = -\sqrt{3}\,iz$ and $y = 0$. The eigenspace for λ_3 is then

$$N(A-(1-\sqrt{3}\,i)I) = \left\{ \begin{bmatrix} -\sqrt{3}\,iz \\ 0 \\ z \end{bmatrix} \mid z \in \mathbb{R} \right\} = \left\{ z \begin{bmatrix} -\sqrt{3}\,i \\ 0 \\ 1 \end{bmatrix} \middle| \, z \in \mathbb{R} \right\},$$

yielding the representative eigenvector $v_3 = \begin{bmatrix} -\sqrt{3}\,i \\ 0 \\ 1 \end{bmatrix}$.