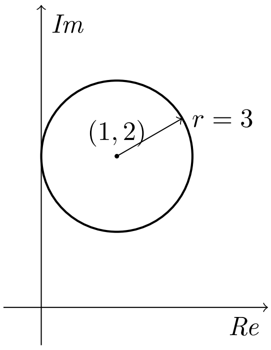
Complex Numbers

Problem Sheet, Wednesday 26.07.2023 (Room U358)

Question 1. *Represent a circle with its center at $(1, 2)$ and a radius of 3 using a complex equation?*

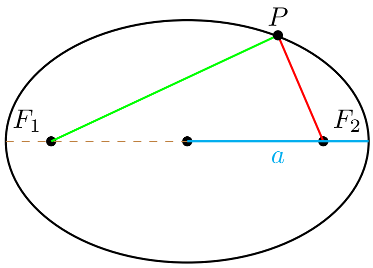


Solution. *We can represent a circle with its center at $(1, 2)$ and a radius of 3 using a complex equation. Let z represent a complex number on the plane, $(1 + 2i)$ represent the center of the circle, and 3 denote the radius of the circle. The equation of the circle is given by:*

$$|z - (1 + 2i)| = 3$$

This equation states that the distance between the complex number z and the center $(1 + 2i)$ is equal to 3, which defines the circle with center $(1, 2)$ and radius 3.

Question 2. *Represent an ellipse with its foci points and semi-major axis using a complex equation? Let F_1 and F_2 represent the foci points and a represents the semi-major axis.*



Solution. *An ellipse can be represented using a complex equation by utilizing the properties of complex numbers and the concept of foci points. The equation involves the distance between a point on the ellipse and the foci points, as well as the semi-major axis.*

To represent an ellipse with foci points F_1 and F_2 and semi-major axis a, the complex equation can be expressed as:

$$|z - F_1| + |z - F_2| = 2a$$

In this equation, z represents a complex number that denotes a point on the ellipse. The term $|z - F_1|$ represents the distance between z and the first focus point F_1, and $|z - F_2|$ represents the distance between z and the second focus point F_2. The sum of these distances is equal to twice the semi-major axis, represented by $2a$.

By plugging in the appropriate coordinates for the foci points and the value of the semi-major axis, the complex equation can be used to represent the desired ellipse.

This complex equation allows us to geometrically describe an ellipse in terms of its foci points and the length of its semi-major axis.

An ellipse can be defined using its foci points.

Question 3. ***Moving the Origin:*** *How can we perform a translation or move the origin in the complex number system?* ***Hint:****Translation refers to the process of moving or shifting an object from one position to another without changing its shape or orientation. In the context of complex numbers, translation involves moving the origin (the reference point) to a new location in the complex plane while maintaining the distances and angles between the complex numbers.*

Solution. *To move the origin in the complex plane, you can add or subtract a complex number from each point. Let's say you want to move P to the origin O.*

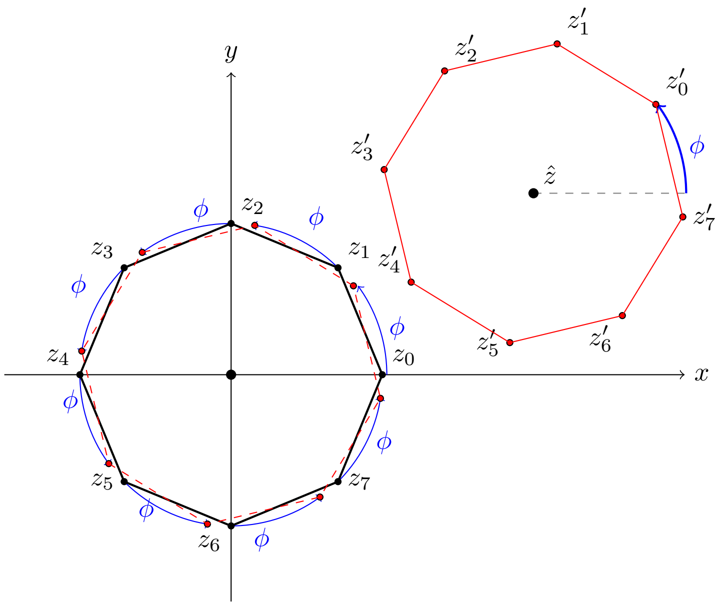
To achieve this, you need to subtract the complex number P from each point in the complex plane. This effectively shifts the entire plane, making P the new origin.

If you have a complex number z representing a point in the original coordinate system, the new position of that point after the origin shift will be $z' = z - P$.

For example, let's consider a complex number $z = 2 + 3i$ in the original coordinate system. If we want to move the origin to $P = 1 + 2i$, the new position of z after the origin shift will be $z' = (2+3i) - (1+2i) = 1 + i$.

In general, to move the origin from O to a new point $P = (a, b)$, you can subtract the complex number P from each point in the complex plane. This will effectively shift the entire plane, making P the new origin.

Question 4. *How would you go about rotating the polygon, obtained by taking the 8th roots of the complex number 1, by an angle of ϕ and then moving it by a vector defined by $\hat{z}$? Can you explain the procedure for both rotation and translation to achieve the desired transformation of the polygon in the complex plane?*



Solution. *Using the following formula:*

$$z^{1/n} = \left\{ \sqrt[n]{r} \left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right) \right) | k = 0, 1, 2, ..., n-1 \right\}$$

we can find all the 8th roots of 1:

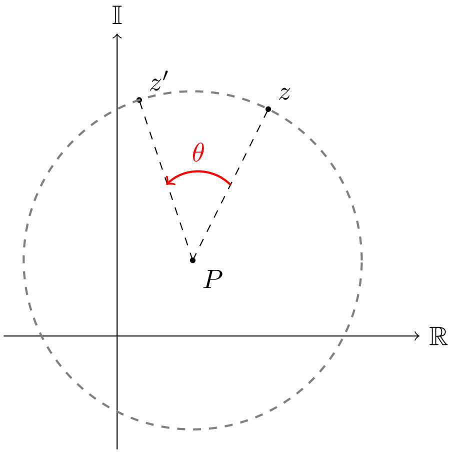
$$
\begin{aligned}
z_0 &= \cos(0) + i\sin(0) = 1 \\
z_1 &= \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) \\
z_2 &= \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i \\
z_3 &= \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) \\
z_4 &= \cos(\pi) + i\sin(\pi) = -1 \\
z_5 &= \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) \\
z_6 &= \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i \\
z_7 &= \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)
\end{aligned}
$$

To rotate the polygon obtained from the 8th roots of the complex number 1 by an angle of ϕ, you can multiply each vertex of the polygon by $e^{i\phi}$. This rotation operation will apply the same angle of rotation to each vertex, resulting in the rotated polygon. To move the rotated polygon by a vector defined by $\hat{z}$, you can add $\hat{z}$ to each vertex of the rotated polygon. This translation operation will shift the entire polygon by the given vector, effectively moving it to the desired location. By combining the rotation and translation operations, you can achieve the desired transformation of the polygon in the complex plane.

$$
\begin{aligned}
z_0' &= \hat{z} + e^{i\phi} z_0 = \hat{z} + \cos(0+\phi) + i\sin(0+\phi) \\
z_1' &= \hat{z} + e^{i\phi} z_1 = \hat{z} + \cos\left(\frac{\pi}{4}+\phi\right) + i\sin\left(\frac{\pi}{4}+\phi\right) \\
z_2' &= \hat{z} + e^{i\phi} z_2 = \hat{z} + \cos\left(\frac{\pi}{2}+\phi\right) + i\sin\left(\frac{\pi}{2}+\phi\right) \\
z_3' &= \hat{z} + e^{i\phi} z_3 = \hat{z} + \cos\left(\frac{3\pi}{4}+\phi\right) + i\sin\left(\frac{3\pi}{4}+\phi\right) \\
z_4' &= \hat{z} + e^{i\phi} z_4 = \hat{z} + \cos(\pi+\phi) + i\sin(\pi+\phi) \\
z_5' &= \hat{z} + e^{i\phi} z_5 = \hat{z} + \cos\left(\frac{5\pi}{4}+\phi\right) + i\sin\left(\frac{5\pi}{4}+\phi\right) \\
z_6' &= \hat{z} + e^{i\phi} z_6 = \hat{z} + \cos\left(\frac{3\pi}{2}+\phi\right) + i\sin\left(\frac{3\pi}{2}+\phi\right) \\
z_7' &= \hat{z} + e^{i\phi} z_7 = \hat{z} + \cos\left(\frac{7\pi}{4}+\phi\right) + i\sin\left(\frac{7\pi}{4}+\phi\right)
\end{aligned}
$$

Question 5. ***Rotating a Complex Number around a Point:***

How would you rotate a complex number z by an angle θ around a fixed point P. Here's a picture illustrating the original position of the complex number z and the rotated position z':



Solution. ***Rotating a Complex Number around a Point***

To rotate a complex number z by an angle θ around a fixed point $P = (a, b)$, follow these steps:

1. Translate the complex plane by subtracting the coordinates of point P from both z and P. This moves the origin of the complex plane to point P. Let $w = z - P$ be the translated complex number.

2. Perform the rotation by multiplying the translated complex number w by the complex number representing the desired rotation, $e^{i\theta}$. Let $w' = w \cdot e^{i\theta}$ be the rotated complex number.

3. Translate the complex plane back to its original position by adding the coordinates of point P to the rotated complex number w'. Let $z' = w' + P$ be the final position of the complex number after rotation.

The complex number z' represents the result of rotating the complex number z by an angle θ around the point P.

It's important to note that the rotation is performed counterclockwise in the complex plane. To rotate clockwise, you can multiply by $e^{-i\theta}$ instead.

Let's consider the complex number $z = 2 + 3i$ and we want to rotate it by an angle of $\frac{\pi}{4}$ around the point $P = (1, 1)$.

To rotate the complex number z around point P, we follow these steps:

1. Translate the complex plane by subtracting the coordinates of point P from both z and P:

$$w = z - P = (2 + 3i) - (1 + 1i) = 1 + 2i$$

2. Perform the rotation by multiplying the translated complex number w by the complex number representing the desired rotation:

$$w' = w \cdot e^{i\frac{\pi}{4}}$$

Using the exponential form $e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, we can calculate:

$$w' = (1 + 2i) \cdot \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = -\frac{\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$$

3. Translate the complex plane back to its original position by adding the coordinates of point P to the rotated complex number w':

$$z' = w' + P = \left(-\frac{\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i\right) + (1 + 1i) = \frac{2 - \sqrt{2}}{2} + \frac{2 + 3\sqrt{2}}{2}i$$

The complex number z' represents the result of rotating the complex number z by an angle of $\frac{\pi}{4}$ around the point P.