Null Spaces

Problem Sheet, Friday 11.08.2023 (Room U358)

Question: Find the solution set for the following system of linear equations :

$$1)\begin{cases} x-y+2z=2 \\ 2x-2y+4z=4 \\ 3x-3y+6z=8 \end{cases}$$

Solution : The system can be compactly represented using an augmented matrix. The matrices A, X, and B are defined as follows:

$$A=\begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{bmatrix}, \quad X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B=\begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}.$$

We then transform the augmented matrix A to its reduced row-echelon form, U:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 2 & -2 & 4 & 4 \\ 3 & -3 & 6 & 8 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

As you can see, after applying Gaussian elimination, the reduced row-echelon form has a row $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 2 \end{array}\right]$. This implies that the system is inconsistent and has no solution. In geometric terms, the three planes defined by these equations do not intersect at a common point in $3D$ space, leading to the lack of a solution.

$$2)\begin{cases} x+2y+3z=2 \\ 2x+3y+z=-3 \\ 3x+y+2z=7 \end{cases}$$

Solution : The system can be compactly represented using an augmented matrix. The matrices A, X, and B are defined as follows:

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}, \quad X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B=\begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}.$$

We then transform the augmented matrix A to its reduced row-echelon form, U:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 2 & 3 & 1 & -3 \\ 3 & 1 & 2 & 7 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & -5 & -7 \\ 0 & 0 & 18 & 36 \end{array}\right]$$

Here, the rank of matrix A is equal to the rank of its reduced row-echelon form. Hence, $rank(A)=3$, which implies that the nullity of A is 0. It means that the only solution to the homogeneous system of equations $AX=0$ is the trivial solution, which is the zero vector. It implies that the system of equations is consistent and has a unique solution: $x=2$, $y=-3$ and $z=2$.

$$3)\begin{cases}2x+3y-z=5\\ 4x-6y+2z=10\\ 6x+9y-3z=15\end{cases}$$

Solution : The system can be compactly represented using an augmented matrix. The matrices A, X, and B are defined as follows:

$$A=\begin{bmatrix}2&3&-1\\4&-6&2\\6&9&-3\end{bmatrix},\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix},\quad B=\begin{bmatrix}5\\10\\15\end{bmatrix}.$$

We then transform the augmented matrix A to its reduced row-echelon form, U:

$$\left[\begin{array}{ccc|c}2&3&-1&5\\4&-6&2&10\\6&9&-3&15\end{array}\right]\Rightarrow\left[\begin{array}{ccc|c}2&3&-1&5\\0&-12&4&0\\0&0&0&0\end{array}\right]$$

Here, the rank of matrix A is equal to the rank of its reduced row-echelon form. Hence, $rank(A)=2$, which implies that the nullity of A (the dimension of the null space) is 1. In this system, x and y are pivot variables, while z is a free variable. Thus, z can take any value, while x and y are determined by z.

A particular solution for $AX=B$, denoted as X_p, can be found by setting $z=3$:

$$\left[\begin{array}{ccc|c}2&3&-1&5\\0&-12&4&0\\0&0&0&0\end{array}\right]\Rightarrow\begin{cases}2x+3y-z=5\\-12y+4z=0\end{cases}\xRightarrow{z=3}\begin{cases}x=2.5\\y=1\end{cases},$$

which gives $X_p=\begin{bmatrix}2.5\\1\\3\end{bmatrix}$. The null space of A, $N(A)$ or equivalently $N(U)$, is defined as the set of vectors X that satisfy $UX=0$:

$$N(A)=N(U)=\left\{X\in\mathbb{R}^3|UX=0\right\}$$

This leads us to solve the system

$$\begin{bmatrix}2&3&-1\\0&-12&4\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

which simplifies to $2x+3y-z=0$ and $-12y+4z=0$, and hence to $x=0$ and $z=3y$.

The vector $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ will lie in $N(U)$ if $x=0$ and $z=3y$, with no restrictions on y. Thus,

$$N(A)=\left\{\begin{bmatrix}0\\y\\3y\end{bmatrix}\middle|\, y\in\mathbb{R}\right\}$$

We can express this in terms of a basis for $N(A)$:

$$N(A)=\left\{y\begin{bmatrix}0\\1\\3\end{bmatrix}\middle|\, y\in\mathbb{R}\right\}$$

Hence, $\begin{bmatrix}0\\1\\3\end{bmatrix}$ forms a basis for $N(A)$.

The solution set of the system $AX = B$ is the sum of the particular solution and the null space, given by $X = X_p + N(A)$.

$$X_p + N(A) = \begin{bmatrix} 2.5 \\ 1 \\ 3 \end{bmatrix} + \left\{ a \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \middle| a \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 2.5 \\ 1 \\ 3 \end{bmatrix} + a \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \middle| a \in \mathbb{R} \right\}$$

Therefore, the solution set consists of the particular solution X_p and all vectors in the null space of A.