Gaussian Elimination

Problem Sheet, Friday 04.08.2023 (Room U358)

Question 1:

Solve the following systems of linear equations using Gaussian elimination:

$$\begin{cases} 2x + y - z + w = 4, \\ x - 3y + 2z - w = -6, \\ 3x + 2y - 2z + 4w = 5, \\ 4x + y + 3z - 2w = 5. \end{cases}$$

Solution: We will use Gaussian elimination to transform the augmented matrix of this system into row-echelon form.

Step 1: The augmented matrix of the system is:

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 4 \\ 1 & -3 & 2 & -1 & -6 \\ 3 & 2 & -2 & 4 & 5 \\ 4 & 1 & 3 & -2 & 5 \end{array}\right]$$

Step 2: We'll use the first row as a pivot row and perform row operations to eliminate the coefficients below the pivot. We'll subtract $\frac{1}{2}$ times the first row from the second row, $\frac{3}{2}$ times the first row from the third row, and 2 times the first row from the fourth row.

$$-\frac{1}{2}\, row_1 + row_2 \quad \left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 4 \\ 1 & -3 & 2 & -1 & -6 \\ 3 & 2 & -2 & 4 & 5 \\ 4 & 1 & 3 & -2 & 5 \end{array}\right] = \left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 4 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{3}{2} & -8 \\ 3 & 2 & -2 & 4 & 5 \\ 4 & 1 & 3 & -2 & 5 \end{array}\right]$$

$$-\frac{3}{2}\, row_1 + row_3 \quad \left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 4 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{3}{2} & -8 \\ 3 & 2 & -2 & 4 & 5 \\ 4 & 1 & 3 & -2 & 5 \end{array}\right] = \left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 4 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{3}{2} & -8 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{5}{2} & -1 \\ 4 & 1 & 3 & -2 & 5 \end{array}\right]$$

$$-2\, row_1 + row_4 \quad \left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 4 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{3}{2} & -8 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{5}{2} & -1 \\ 4 & 1 & 3 & -2 & 5 \end{array}\right] = \left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 4 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{3}{2} & -8 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{5}{2} & -1 \\ 0 & -1 & 5 & -4 & -3 \end{array}\right]$$

Step 3: We'll use the second row as a pivot row and perform row operations to eliminate the coefficients below the pivot. We'll add $\frac{1}{7}$ times the second row to the third row and $-\frac{2}{7}$ times the second row to the fourth row.

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 4 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{3}{2} & -8 \\ 0 & 0 & -\frac{1}{7} & \frac{16}{7} & -\frac{15}{7} \\ 0 & 0 & \frac{30}{7} & -\frac{25}{7} & -\frac{5}{7} \end{array}\right]$$

Step 4: We'll use the third row as a pivot row and perform row operations to eliminate the coefficient below the pivot.

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 4 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{3}{2} & -8 \\ 0 & 0 & -\frac{1}{7} & \frac{16}{7} & -\frac{15}{7} \\ 0 & 0 & 0 & 65 & -65 \end{array}\right]$$

The resulting matrix is now in row-echelon form, with the pivots highlighted in red. After performing Gaussian elimination and reaching the row-echelon form, we can now utilize back substitution for the following system of equations to find the values of the variables.

$$\begin{cases} 2x + y - z + w = 4, \\ -\frac{7}{2}y + \frac{5}{2}z - \frac{3}{2}w = -8, \\ -\frac{1}{7}z + \frac{16}{7}w = -\frac{15}{7}, \\ 65w = -65. \end{cases}$$

The solution to the system of equations is $x = 1$, $y = 2$, $z = -1$, and $w = -1$.

Question 2:

Find the inverse of the following matrix A using Gauss-Jordan elimination:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 2 \\ 1 & 0 & 5 \end{bmatrix}$$

Solution: We will find A^{-1} using Gauss-Jordan elimination:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 2 \\ 1 & 0 & 5 \end{bmatrix}$$

To find the inverse of A, since $AA^{-1} = I$ we can augment A with the identity matrix I:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 4 & 2 & 0 & 1 & 0 \\ 1 & 0 & 5 & 0 & 0 & 1 \end{array}\right]$$

Next, we apply elementary row operations to transform the augmented matrix into reduced row echelon form.

1. Multiply the first row by 1/2 to create a leading 1:

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 4 & 2 & 0 & 1 & 0 \\ 1 & 0 & 5 & 0 & 0 & 1 \end{array}\right]$$

2. Subtract the first row from the third row:

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 4 & 2 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & \frac{7}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right]$$

3. Multiply the second row by $\frac{1}{4}$ to create a leading 1:

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & \frac{7}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right]$$

4. Add $\frac{1}{2}$ times the second row to the third row:

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{15}{4} & -\frac{1}{2} & \frac{1}{8} & 1 \end{array}\right]$$

5. Multiply the third row by $\frac{4}{15}$ to create a leading 1:

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & -\frac{2}{15} & \frac{1}{30} & \frac{4}{15} \end{array}\right]$$

6. Subtract $\frac{1}{2}$ times the third row from the second row:

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{15} & \frac{7}{30} & -\frac{2}{15} \\ 0 & 0 & 1 & -\frac{2}{15} & \frac{1}{30} & \frac{4}{15} \end{array}\right]$$

7. Subtract $\frac{3}{2}$ times the third row from the first row:

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{7}{10} & -\frac{1}{20} & -\frac{2}{5} \\ 0 & 1 & 0 & \frac{1}{15} & \frac{7}{30} & -\frac{2}{15} \\ 0 & 0 & 1 & -\frac{2}{15} & \frac{1}{30} & \frac{4}{15} \end{array}\right]$$

8. Subtract $\frac{1}{2}$ times the second row from the first row:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{15} & \frac{7}{30} & -\frac{2}{15} \\ 0 & 0 & 1 & -\frac{2}{15} & \frac{1}{30} & \frac{4}{15} \end{array}\right]$$

Thus, the inverse of matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 2 \\ 1 & 0 & 5 \end{bmatrix}$ is given by:

$$A^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ \frac{1}{15} & \frac{7}{30} & -\frac{2}{15} \\ -\frac{2}{15} & \frac{1}{30} & \frac{4}{15} \end{bmatrix} = \frac{1}{30} \begin{bmatrix} 20 & -5 & -10 \\ 2 & 7 & -4 \\ -4 & 1 & 8 \end{bmatrix}$$