Matrices

Problem Sheet, Monday 31.07.2023 (Room U358)

Elementary Matrices

An elementary matrix is a unique category of square matrix which, upon left multiplication to a given matrix, administers one of these elementary row transformations:

1. Swapping two specific rows,
2. Multiplying a single row by a non-zero scalar,
3. Adding a scaled version of one row to another row.

These operations are pivotal in linear algebra, playing critical roles in the calculation of a matrix's inverse or determinant, and in solving systems of linear equations.

Below are some exercises designed to enhance your understanding and application of elementary matrices:

Question 1: Row Exchange

Design an elementary matrix E that, upon matrix multiplication to a 4×4 matrix, interchanges the 2nd and 4th rows. Implement this operation and subsequently find the reverse of E by reversing this transformation. Examine the results and derive your conclusions.

$$EA = E \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}.$$

Solution

To design the elementary matrix E that interchanges the 2nd and 4th rows upon matrix multiplication, we can start with the identity matrix I and make the necessary modifications. Let E be defined as:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

To implement the row interchange operation, we can multiply matrix E with a 4×4 matrix A:

$$EA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$$

To reverse the row interchange operation, we need to interchanges the 2nd and 4th rows again:

$$E^{-1} = E.$$

Question 2:Row Scaling by a Nonzero Scalar

Design an elementary matrix E that, upon matrix multiplication to a 4×4 matrix, scales the 3rd row by a factor of 5. Execute this operation and subsequently find the reverse of E. Analyse the results and infer your conclusions.

$$EA = E \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 5a_{31} & 5a_{32} & 5a_{33} & 5a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Solution

The elementary matrix E is designed to scale the 3rd row of a 4×4 matrix by a factor of 5. It can be expressed as:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, we can perform the matrix multiplication of E with a given 4×4 matrix A:

$$EA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 5a_{31} & 5a_{32} & 5a_{33} & 5a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

To reverse the transformation performed by matrix E, since matrix E scales the 3rd row by 5, matrix E^{-1} must scale it back by dividing by 5:

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Question 3: Addition of a Scaled Version of One Row to Another Row

Design an elementary matrix E that, upon matrix multiplication to a 4×4 matrix, adds three times the 1st row to the 2nd row. Perform this operation and subsequently reverse it to find E^{-1}. Review the outcomes and formulate your conclusions.

$$EA = E \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 3a_{11}+a_{21} & 3a_{12}+a_{22} & 3a_{13}+a_{23} & 3a_{14}+a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Solution

To add three times the 1st row to the 2nd row, we modify the entries of I accordingly:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, we can perform the matrix multiplication of E with a given 4×4 matrix A:

$$EA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 3a_{11}+a_{21} & 3a_{12}+a_{22} & 3a_{13}+a_{23} & 3a_{14}+a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

To find the inverse of E, we need to perform the reverse operation. By subtracting three times the 1st row from the 2nd row, we obtain:

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Consequently, E^{-1} is the elementary matrix that reverses the original operation.

Question 4:

Based on the elementary matrices we obtained from the previous question, let's consider multiplying each of these elementary matrices with a 4×4 matrix from the right AE. What transformations do you anticipate these matrices will induce? Please explain your thoughts and provide examples or reasoning to support your answer."

Solution

Consider the following 4×4 matrix, A:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

Let's E be the elementary matrix in Question 1, that swaps the 2nd and 4th rows of matrix A:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

To apply this row operation to matrix A, we perform the matrix multiplication EA:

$$EA = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 13 & 14 & 15 & 16 \\ 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

As we can see, the 2nd and 4th rows of matrix A have been interchanged.
Now, we calculate the matrix multiplication AE:

$$AE = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 5 & 8 & 7 & 6 \\ 9 & 12 & 11 & 10 \\ 13 & 16 & 15 & 14 \end{bmatrix}$$

As we can see, the 2nd and 4th columns of matrix A have been interchanged. Thus, the order of multiplication between matrices is significant, as demonstrated in the example.