Diagonalisation

Problem Sheet, Monday 21.08.2023 (Room U119)

Question 1: Given the symmetric matrix:

$$C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

we are tasked to:

1. Find the eigenvalues and eigenvectors of C.
2. Determine if C is diagonalizable.
3. Find an orthonormal matrix P such that $C = PDP^T$, where D is a diagonal matrix.
4. Determine the determinant of C.
5. Compute C^{1000}.

Solution :

Preliminary: Since C is a real symmetric matrix, it is guaranteed to be diagonalizable.

(i) Given eigenvalues of C are:

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = -1$$

The corresponding eigenvectors are:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}.$$

Normalized, these vectors become:

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}.$$

Thus, the orthonormal matrix is:

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$$

with the diagonal matrix:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(ii) As previously established, C is diagonalizable.

(iii) Verification of $C = PDP^T$:

$$PDP^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} = C$$

(iv) Since P is orthonormal, $P^T = P^{-1}$. Thus, the determinant of C is:

$$\det(C) = \det(PDP^T) = \det(P)\det(D)\det(P^T) = \det(D) = -2$$

(v) To compute C^{1000}, we observe that:

$$C^2 = PD(P^TP)DP^T = PD^2P^T$$

Using this logic inductively:

$$C^{1000} = PD^{1000}P^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1^{1000} & 0 & 0 \\ 0 & 2^{1000} & 0 \\ 0 & 0 & (-1)^{1000} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}$$

Question 2: Given the symmetric matrix:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

Find an orthonormal matrix P such that $A = PDP^T$, where D is a diagonal matrix.

Solution : Observe that A is a real symmetric matrix. By the above theorem, we know that A is diagonalizable. i.e. we will be able to find a sufficient number of linearly independent eigenvectors. The eigenvalues of A were: $\lambda_1 = 1$, $\lambda_2 = 4$. We found two linearly independent eigenvectors corresponding to $\lambda_1 = 1$:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

And one eigenvector corresponding to $\lambda_2 = 4$:

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Since A is a real symmetric matrix, eigenvectors corresponding to distinct eigenvalues are orthogonal. However, the eigenvectors corresponding to eigenvalue $\lambda_1 = 1$, $\mathbf{v}_1$ and $\mathbf{v}_2$ are not orthogonal to each other. We have to use the Gram Schmidt process.

Using the Gram Schmidt process:

$$\mathbf{u}_1 = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{w}_2 = \mathbf{v}_2 - (\mathbf{v}_2.\mathbf{u}_1)\mathbf{u}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{\mathbf{w_2}}{\|\mathbf{w_2}\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Normalize the vectors: $\mathbf{u_1}$ and $\mathbf{u_2}$ are normalised but we need to normalise $\mathbf{v_3}$:

$$\mathbf{u}_3 = \frac{\mathbf{v_3}}{\|\mathbf{v_3}\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Now we have a set of orthonormal eigenvectors corresponding to eigenvalues $\{1, 1, 4\}$ as follows:

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Thus we have an orthonormal matrix:

$$P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

and the corresponding diagonal matrix

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Verification $A = PDP^T$:

$$PDP^T = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} = A$$