Determinant

Problem Sheet, Monday 14.08.2023 (Room U358)

Question 1: An *orthogonal matrix* is a square matrix with real entries such that its transpose is also its inverse. Formally, for an orthogonal matrix A, we have:

$$A^TA = AA^T = I$$

where A^T denotes the transpose of A and I is the identity matrix of the same order as A. Thus the inverse of an orthogonal matrix is its transpose.

The columns (and rows) of an orthogonal matrix form an orthonormal set, implying that they are mutually orthogonal and each has unit length.

1. **Preservation of Dot Product**: Prove that if A is orthogonal, then for any vectors x and y in $\mathbb{R}^n$,

$$x.y = (Ax).(Ay)$$

This property indicates that the orthogonal transformation preserves the angles between vectors.

Solution:

$$\begin{aligned}(Ax).(Ay) &= (Ax)^T(Ay)\\ &= x^TA^TAy\\ &= x^TIy\\ &= x^Ty\\ &= x.y\end{aligned}$$

2. **Determinant Value**: Prove that the determinant of an orthogonal matrix is either $+1$ or -1, and specify what each value corresponds to in geometric terms.

Solution: Using the property $A^TA = I$, we have:

$$\det(A^TA) = \det(I) \implies \det(A^T)\det(A) = 1$$

Since $\det(A^T) = \det(A)$:

$$(\det(A))^2 = 1$$

This implies $\det(A)$ is either $+1$ or -1. Specifically:

- A determinant of $+1$ corresponds to a rotation.
- A determinant of -1 corresponds to a reflection.

3. **Preservation of Lengths**: Prove that for any vector x in $\mathbb{R}^n$ and an orthogonal matrix A, the lengths before and after transformation are preserved: $||Ax|| = ||x||$.

Solution:

$$\begin{aligned}||Ax|| &= \sqrt{(Ax)\cdot(Ax)}\\ &= \sqrt{(Ax)^TAx}\\ &= \sqrt{x^TA^TAx}\\ &= \sqrt{x^Tx}\\ &= \sqrt{x.x}\\ &= ||x||\end{aligned}$$

Orthogonal matrices are pivotal in diverse areas of mathematics, especially in linear algebra, geometry, and signal processing, with applications prominently seen in rotations and reflections in space.

4. **Orthogonal Matrix Form an Orthonormal Set**: Prove that the columns (and by symmetry, rows) of an orthogonal matrix form an orthonormal set. Specifically, demonstrate the following:

(a) They are mutually orthogonal.

(b) Each has unit length.

Solution

Let A be an $n \times n$ orthogonal matrix. By the definition of an orthogonal matrix, we know that:

$$A^T A = AA^T = I$$

Let's denote the columns of A as $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$.

Proof of Mutual Orthogonality: Consider the i-th and j-th columns, $\mathbf{a}_i$ and $\mathbf{a}_j$ respectively. The dot product of these columns can be represented as the i, j-th entry of the matrix $A^T A$:

$$\mathbf{a}_i \cdot \mathbf{a}_j = (A^T A)_{ij}$$

Since $A^T A = I$, the i, j-th entry of this product is 0 for $i \neq j$ and 1 for $i = j$. This means the columns are mutually orthogonal.

Proof of Unit Length: For any column $\mathbf{a}_i$:

$$\begin{aligned} \|\mathbf{a}_i\| &= \sqrt{\mathbf{a}_i \cdot \mathbf{a}_i} \\ &= \sqrt{(A^T A)_{ii}} \\ &= 1 \end{aligned}$$

This is because the diagonal entries of the identity matrix are all 1s.

Thus, the columns of A are mutually orthogonal and each has unit length. By symmetry, the same applies to the rows. Therefore, the columns (and rows) of an orthogonal matrix form an orthonormal set.

Question 2: Prove that a transformation represented by a 2×2 matrix A is area-preserving if and only if $|\det(A)| = 1$.

Given the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

determine whether the transformation it represents is area-preserving.

Solution. Part 1: Prove that a transformation represented by matrix A is area-preserving if and only if $|\det(A)| = 1$.

Proof: Given two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$, the area A of the parallelogram spanned by these vectors is given by the absolute value of the determinant of the matrix whose columns (or rows) are these vectors:

$$A = \left|\det \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}\right|$$

When $\mathbf{u}$ and $\mathbf{v}$ are transformed by matrix A to $A\mathbf{u}$ and $A\mathbf{v}$ respectively, the new area A' is:

$$A' = \left|\det \begin{bmatrix} A\mathbf{u} & A\mathbf{v} \end{bmatrix}\right|$$

But $\begin{bmatrix} A\mathbf{u} & A\mathbf{v} \end{bmatrix} = A \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}$, then

$$A' = \left|\det \begin{bmatrix} A\mathbf{u} & A\mathbf{v} \end{bmatrix}\right| = \left|\det(A) \det \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}\right|$$

For the transformation to be area-preserving, we require $A' = A$. Hence,

$$\left|\det(A) \det \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}\right| = \left|\det \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}\right|$$

This must hold for all vectors $\mathbf{u}$ and $\mathbf{v}$. The only way this can be true is if $|\det(A)| = 1$. Thus, an area-preserving transformation represented by matrix A occurs if and only if $\det(A) = 1$ or $\det(A) = -1$.

Part 2: Given matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Calculate:

$$\det(A) = 2 \times 1 - 1 \times 1 = 1$$

Since $\det(A) = 1$, the transformation represented by matrix A is area-preserving.

Question 3: Suppose A is an $n \times n$ matrix and B is the matrix derived from A by interchanging any two rows (or columns). Prove that $\det(B) = -\det(A)$.

Solution: Consider an $n \times n$ matrix A.

Let E be the elementary matrix obtained by interchanging two rows, say i and j, of the identity matrix I_n. In other words, E has all diagonal entries as 1 except for the i^{th} and j^{th} rows, which come from the j^{th} and i^{th} rows of I_n, respectively.

Now, EA is the matrix obtained by swapping the i^{th} and j^{th} rows of A. Let's call this matrix B. Therefore, $B = EA$.

To see the effect on determinants, consider the determinant of E:

Consider the elementary row swapping matrix E that swaps row i and row j of an identity matrix I. Then in row i the only nonzero is in column j and in row j the only nonzero is in column i. Without loss of generality, let's consider $i < j$. For example, if we consider $i = 1$ and $j = 3$. Then E is:

$$E = \begin{bmatrix} 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Expanding the determinant along the row i, we have:

$$\det(E) = (-1)^{i+j} \times 1 \times \det(E_i) = (-1)^{i+j} \times \det(E_i)$$

Where E_i is the submatrix of E obtained by removing the row i and column j. Now to indicate $\det(E_i)$ we expand the determinant along the row $j-1$, we have:

$$\det(E_i) = (-1)^{i+j-1} \times 1 \times \det(E_j) = (-1)^{i+j-1} \times \det(E_j)$$

Where E_j is the submatrix of E_i obtained by removing the row $j-1$ and column i. E_j is essentially an identity matrix of size $n-2$, and thus its determinant is 1.

Therefore:

$$\det(E) = (-1)^{i+j} \times \det(E_i) = (-1)^{i+j}(-1)^{i+j-1} \times \det(E_j) = (-1)^{i+j}(-1)^{i+j-1} = (-1)^{2(i+j)-1} = -1$$

This proves that the elementary row swapping matrix has a determinant of -1.

$$\det(E) = -1$$

Using properties of determinants, we have:

$$\det(EA) = \det(E) \times \det(A) \implies \det(B) = -\det(A)$$

Thus, the determinant of the matrix changes sign when two of its rows are interchanged.