Span and Linear Independence

Problem Sheet, Monday 7.08.2023 (Room U358)

Question 1: The Hilbert matrix is a specific type of matrix where the (i, j)-th entry is given by $1/(i+j-1)$. Consider the 4 × 4 Hilbert matrix, denoted by A:

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}$$

Demonstrate that the rows of the matrix A are linearly independent.

Solution : The LU factorization of the Hilbert matrix gives us:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{3} & 1 & 1 & 0 \\ \frac{1}{4} & \frac{9}{10} & \frac{3}{2} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{3}{40} \\ 0 & 0 & \frac{1}{180} & \frac{1}{120} \\ 0 & 0 & 0 & \frac{1}{2800} \end{bmatrix}$$

In the case of the Hilbert matrix, the matrix U has no zero rows. This implies that the original matrix A has linearly independent rows. Therefore, the rows of the Hilbert matrix are linearly independent.

Question 2: Given the following vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 4 \\ 6 \\ 2 \\ 0 \end{bmatrix},$$

Your challenge is to find a basis for the space spanned by these vectors.

Solution : First, we form the matrix A by treating the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ as rows:

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 1 & 0 \\ 3 & 5 & 2 & 1 \\ 4 & 6 & 2 & 0 \end{bmatrix}$$

Then, we perform the LU factorization on this matrix to find L and U. After computing, we obtain:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Observe that the rows of U that are non-zero vectors are $\mathbf{u}_1 = [1, 2, 1, 1]^T$ and $\mathbf{u}_2 = [0, -1, -1, -2]^T$.

So, the basis for the span of $\mathbf{v}_1, \mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ is $\mathbf{u}_1$, $\mathbf{u}_2$.