Vectors

Problem Sheet, Friday 28.07.2023 (Room U358)

Question 1. *For vectors* $\mathbf{a}$ *and* $\mathbf{b}$*, it is given that* $|\mathbf{a}+\mathbf{b}| = 5$ *and* $|\mathbf{a}-\mathbf{b}| = 3$*. Compute the inner product* $\mathbf{a}\cdot\mathbf{b}$*.*

Solution. *Let's first square both sides of the given equations:*

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = 5^2 \quad \textit{and} \quad (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = 3^2$$

Expanding the dot products and simplifying, we have:

$$\|\mathbf{a}\|^2 + 2(\mathbf{a}\cdot\mathbf{b}) + \|\mathbf{b}\|^2 = 25 \quad \textit{and} \quad \|\mathbf{a}\|^2 - 2(\mathbf{a}\cdot\mathbf{b}) + \|\mathbf{b}\|^2 = 9$$

Subtracting the second equation from the first, we get:

$$4(\mathbf{a}\cdot\mathbf{b}) = 16$$

Therefore, $\mathbf{a}\cdot\mathbf{b} = \frac{16}{4} = 4$.

Question 2. *Consider three vectors* $\mathbf{u}$*,* $\mathbf{v}$*, and* $\mathbf{w}$ *in* $\mathbb{R}^3$ *defined as follows:*

$$\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}.$$

a) Calculate the vector $\mathbf{u}\times\mathbf{v}$*.*

b) Determine whether the vector $\mathbf{u}\times\mathbf{v}$ *is orthogonal to* $\mathbf{w}$*.*

c) Find a vector that is orthogonal to both $\mathbf{u}$ *and* $\mathbf{w}$*.*

Solution. *a) To calculate the cross product* $\mathbf{u}\times\mathbf{v}$*, we use the formula:*

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \mathbf{i}(u_2v_3 - u_3v_2) - \mathbf{j}(u_1v_3 - u_3v_1) + \mathbf{k}(u_1v_2 - u_2v_1)$$

Expanding the determinant, we have:

$$\mathbf{u}\times\mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} = \begin{bmatrix} (-1)(-2) - (3)(0) \\ (3)(1) - (2)(-2) \\ (2)(0) - (-1)(1) \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix}.$$

Therefore, $\mathbf{u}\times\mathbf{v} = \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix}$.

b) To determine whether $\mathbf{u}\times\mathbf{v}$ *is orthogonal to* $\mathbf{w}$*, we calculate their dot product and check if it equals zero.*

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = (2)(3) + (7)(1) + (1)(-4) = 6 + 7 - 4 = 9 \neq 0.$$

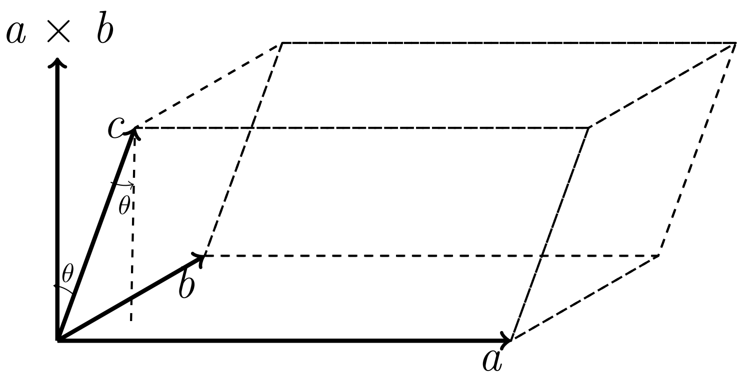
Since $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} \neq 0$*, we conclude that* $\mathbf{u}\times\mathbf{v}$ *is not orthogonal to* $\mathbf{w}$*.*

c) To find a vector orthogonal to both $\mathbf{u}$ *and* $\mathbf{w}$*, we can take their cross product.*

$$\mathbf{u}\times\mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} (-1)(-4) - (3)(1) \\ (3)(3) - (2)(-4) \\ (2)(1) - (-1)(3) \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \\ 5 \end{bmatrix}.$$

Therefore, $\begin{bmatrix} 1 \\ 17 \\ 5 \end{bmatrix}$ *is a vector that is orthogonal to both* $\mathbf{u}$ *and* $\mathbf{w}$*.*

Question 3. *Consider a parallelepiped in three-dimensional space with three adjacent edges defined by the vectors* $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$, *and* $\mathbf{c} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$. *Find the volume of the parallelepiped using the dot and cross products.*



Solution. *To find the volume* V *of the parallelepiped, we first calculate the area of the parallelogram formed by* $\mathbf{a}$ *and* $\mathbf{b}$ *and then multiply it by the height of the parallelepiped, which is equal to* $|\mathbf{c}|\cos(\theta)$, *where* θ *is the angle between* $\mathbf{c}$ *and* $\mathbf{a} \times \mathbf{b}$. *The area of the parallelogram formed by* $\mathbf{a}$ *and* $\mathbf{b}$ *is given by* $|\mathbf{a} \times \mathbf{b}|$.

$$V = |\mathbf{a} \times \mathbf{b}||\mathbf{c}|\cos(\theta) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}) \cdot \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ -9 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = (-14) + (30) - 9 = 7$$

Therefore, the volume of the parallelepiped is 7 cubic units.