# Linear Transformation

Problem Sheet, Friday 11.08.2023 (Room U358)

**Question 1:** A projection operator $P$ maps a vector onto a subspace, creating its "shadow" or the closest point in that subspace. This is represented as:

$$P(\mathbf{v}) = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$$

Where $\mathbf{v}$ is the vector being projected, and $\mathbf{u}$ is the vector spanning the subspace. Verify whether the Projection Operator is a linear transformation, and explain what makes it so.

**Answer:** The projection operator is indeed linear, and this can be verified as follows:

- **Additivity** is verified by the equation:

  $$P(\mathbf{v}+\mathbf{w}) = P(\mathbf{v}) + P(\mathbf{w})$$

  This means that the projection of the sum of two vectors is the sum of their individual projections.

- **Homogeneity** is verified by the equation:

  $$P(c\cdot\mathbf{v}) = c\cdot P(\mathbf{v})$$

  This means that the projection of a scaled vector is the same as scaling the projection of the original vector.

**Question 2:** Consider the integral operator $T : C([a,b]) \to \mathbb{R}$, where $C([a,b])$ is the set of continuous functions on the interval $[a,b]$, defined by

$$T(f) = \int_a^b f(x)\,dx$$

for each $f \in C([a,b])$. Determine whether $T$ is a linear transformation.

**Answer:** To check whether $T$ is a linear transformation, we need to verify the two properties of linearity: additivity and homogeneity.

**Additivity**: We need to show that for any functions $f,g \in C([a,b])$, $\quad T(f+g) = T(f)+T(g)$:

$$\begin{aligned}
T(f+g) &= \int_a^b (f(x)+g(x))\,dx \\
&= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \\
&= T(f)+T(g)
\end{aligned}$$

**Homogeneity**: We need to show that for any function $f \in C([a,b])$ and any scalar $c \in \mathbb{R}$, $T(cf) = cT(f)$:

$$\begin{aligned}
T(cf) &= \int_a^b (cf(x))\,dx \\
&= c\int_a^b f(x)\,dx \\
&= cT(f)
\end{aligned}$$

Since the integral operator $T$ satisfies both additivity and homogeneity, it is indeed a linear transformation.

**Question 3:** Verify whether the transformation $T:\mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} 2x-y \\ y+z \end{bmatrix}$ is a linear transformation.

**Answer:** A transformation is linear if it satisfies the following two properties:

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(c\mathbf{u}) = cT(\mathbf{u})$, where $c$ is a scalar.

Let's denote $\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$.

1. We have

$$T(\mathbf{u}+\mathbf{v}) = T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right) = \begin{bmatrix} 2(x_1+x_2)-(y_1+y_2) \\ (y_1+y_2)+(z_1+z_2) \end{bmatrix}$$
$$= \begin{bmatrix} 2x_1-y_1 \\ y_1+z_1 \end{bmatrix} + \begin{bmatrix} 2x_2-y_2 \\ y_2+z_2 \end{bmatrix} = T(\mathbf{u}) + T(\mathbf{v}).$$

2. Now let's check for any scalar $c$:

$$T(c\mathbf{u}) = T\left(\begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}\right) = \begin{bmatrix} 2(cx_1)-cy_1 \\ cy_1+cz_1 \end{bmatrix} = c\begin{bmatrix} 2x_1-y_1 \\ y_1+z_1 \end{bmatrix} = cT(\mathbf{u}).$$

Since $T$ satisfies both properties, it is indeed a linear transformation.

**Question 4:** Consider a basis in $\mathbb{R}^3$, denoted as $B = \{b_1, b_2, b_3\}$, where

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad b_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

A vector $\mathbf{v}$ in $\mathbb{R}^3$ is given in the standard basis as

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

What are the coordinates of this vector in the new basis $B$?

**Answer:** First, we find the transition matrix from the standard basis $E$ to the new basis $B$. This matrix is constructed by placing the basis vectors of $B$ in the columns:

$$P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Next, we calculate the inverse of this matrix:

$$P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

To express vector $\mathbf{v}$ in terms of the basis $B$, we apply the inverse of the transition matrix to $\mathbf{v}$:

$$[v]_B = P^{-1}[v]_E = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 3 \\ -\frac{1}{2} \end{bmatrix}.$$

Therefore, the coordinates of the vector $\mathbf{v}$ in the basis $B$ are $(\frac{3}{2}, 3, -\frac{1}{2})$.

**Question 5:** Let's consider two sets of basis vectors in three-dimensional space ($\mathbb{R}^3$):

Basis $Q$ is represented by vectors:

$$q_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad q_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Basis $R$ is represented by vectors:

$$r_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad r_3 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

Also, consider the vector $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$.

You are asked to:

1. Compute the change-of-basis matrix that transitions from basis $Q$ to basis $R$.
2. Represent the vector $\mathbf{v}$ in the $R$ basis.

**Answer:**

**1. Calculating the Change-of-Basis Matrix from $Q$ to $R$ :** The task here is to express each vector from basis $R$ as a linear combination of vectors from basis $Q$.

Upon doing so, we find:

$$\begin{aligned} r_1 &= \frac{1}{3}q_1 + \frac{1}{3}q_2 + \frac{1}{3}q_3, \\ r_2 &= \frac{1}{9}q_1 + \frac{7}{9}q_2 - \frac{5}{9}q_3, \\ r_3 &= -\frac{4}{9}q_1 + \frac{8}{9}q_2 + \frac{2}{9}q_3, \end{aligned}$$

Consequently, the change-of-basis matrix $P$ from $Q$ to $R$ becomes:

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{9} & -\frac{4}{9} \\ \frac{1}{3} & \frac{7}{9} & \frac{8}{9} \\ \frac{1}{3} & -\frac{5}{9} & \frac{2}{9} \end{bmatrix}$$

The inverse of this matrix $P^{-1}$ (which gives the transition from basis $R$ to basis $Q$) is:

$$P^{-1} = \frac{1}{2} \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix}$$

**2. Representing the Vector $\mathbf{v}$ in the $R$ Basis :** Firstly, express $\mathbf{v}$ in terms of the basis $Q$ to obtain its coordinates $[v]_Q$:

$$[v]_Q = \begin{bmatrix} \frac{1}{3} \\ \frac{7}{3} \\ \frac{1}{3} \end{bmatrix}$$

Subsequently, use the change-of-basis matrix $P^{-1}$ to find the coordinates of $\mathbf{v}$ in terms of the basis $R$ ($[v]_R$):

$$[v]_R = P^{-1}[v]_Q = \frac{1}{2}\begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{3} \\ \frac{7}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Hence, the coordinates of the vector $\mathbf{v}$ in the $R$ basis are 2, 1, and 1.