A = LU

Problem Sheet, Friday 04.08.2023 (Room U358)

Question 1: In the PLU decomposition of a 5 by 5 matrix A, the permutation matrix P is employed to rearrange the rows, resulting in row 1 becoming row 3, row 2 becoming row 4, row 4 becoming row 5, row 5 becoming row 2 and row 3 becoming row 1. To further analyse calculate the inverse of P.

Solution : To calculate the permutation matrix P, it is sufficient to apply the same permutation to the Identity matrix. Additionally, it is well-known that the inverse of a permutation matrix is equal to its transpose.

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad P^{-1} = P^T = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Question 2: Find the LU factorisation of the matrix A.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{bmatrix}$$

Solution : We use Gaussian elimination to find L and U such that $A = LU$. Every elementary row operation can by done by multiplication of an invertible matrix as follows :

1. Subtract 2 times the first row from the second row, by multiplying the following matrix e_1 :

$$e_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -8 \\ 3 & 6 & -3 \end{bmatrix}$$

2. Subtract 3 times the first row from the third row, by multiplying the following matrix e_2

$$e_2(e_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} (e_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -8 \\ 3 & 6 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -8 \\ 0 & -3 & -12 \end{bmatrix}$$

3. Subtract $\frac{3}{2}$ times the second row from the third row using e_3

$$e_3(e_2 e_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{3}{2} & 1 \end{bmatrix} (e_2 e_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -8 \\ 0 & -3 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -8 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore

$$(e_3 e_2 e_1) A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -8 \\ 0 & 0 & 0 \end{bmatrix}. \quad \Rightarrow \quad A = (e_3 e_2 e_1)^{-1} \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -8 \\ 0 & 0 & 0 \end{bmatrix}.$$

Set $U := \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -8 \\ 0 & 0 & 0 \end{bmatrix}$ and $L := (e_3 e_2 e_1)^{-1}$. Let us calculate L :

$$L = (e_3 e_2 e_1)^{-1} = e_1^{-1} e_2^{-1} e_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{3}{2} & 1 \end{bmatrix}$$

Now we can verify the decomposition by multiplying $L$and U:

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -8 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{bmatrix} = A$$

As you can see, the product of L and U gives us the original matrix A, confirming the correctness of the LU decomposition.

Question 3: A Vandermonde matrix is a matrix in which each row is a geometric progression of the form $(1, a, a^2, a^3, \ldots)$. Vandermonde matrices are special matrices with intriguing properties that find applications in various mathematical disciplines. In this question, we will explore the LU factorization of a Vandermonde matrix. Consider the 4×4 Vandermonde matrix A :

$$\begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{bmatrix}.$$

In a LU factorization of matrix A, the L and U are as follows :

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & \frac{c-a}{b-a} & 1 & 0 \\ 1 & \frac{d-a}{b-a} & \frac{(d-a)(d-b)}{(c-b)(c-a)} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & a & a^2 & a^3 \\ 0 & b-a & b^2-a^2 & b^3-a^3 \\ 0 & 0 & (c-b)(c-a) & (c-b)(c-a)(c+b+a) \\ 0 & 0 & 0 & (d-c)(d-b)(d-a) \end{bmatrix}.$$

Use this factorization to solve the following linear system of equations :

$$\begin{cases} x+y+z+w=5 \\ x+2y+4z+8w=19 \\ x-y+z-w=-3 \\ x-2y+4z-8w=-25. \end{cases}$$

Solution : We have a linear system of equation as $AX = B$. For the given system,

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & 4 & -8 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 6 & 12 \\ 0 & 0 & 0 & -12 \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ 19 \\ -3 \\ -25 \end{bmatrix}.$$

The solution process now proceeds in two stages.

First Stage: Solving $LY = B$ for $Y = UX$.

Second Stage: Solving $UX = Y$.

Solving $LY = B$ yields :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 2 & 1 \end{bmatrix} Y = \begin{bmatrix} 5 \\ 19 \\ -3 \\ -25 \end{bmatrix} \quad \Rightarrow \quad Y = \begin{bmatrix} 5 \\ 14 \\ 20 \\ -28 \end{bmatrix}$$

Solving $UX = Y$ gives the solution to the original system of equations :

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 6 & 12 \\ 0 & 0 & 0 & -12 \end{bmatrix} X = \begin{bmatrix} 5 \\ 14 \\ 20 \\ -28 \end{bmatrix} \quad \Rightarrow \quad X = \begin{bmatrix} 7/3 \\ 5/3 \\ -4/3 \\ 7/3 \end{bmatrix}$$

Therefore, the solution to the original system of equations is $x = \frac{7}{3}$, $y = \frac{5}{3}$, $z = -\frac{4}{3}$, and $w = \frac{7}{3}$.