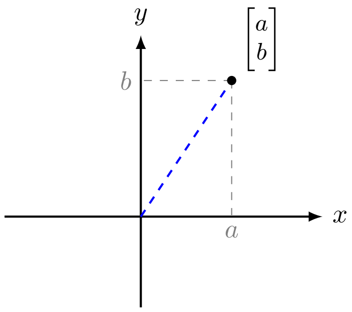
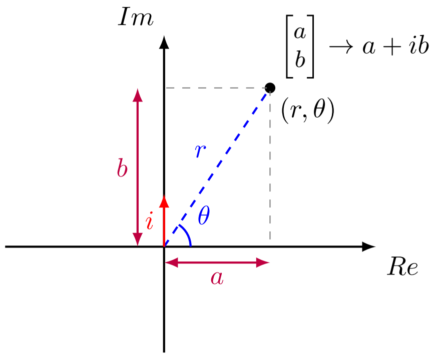
# Complex Numbers

Wednesday 26.07.2023 (Room M1)

## 1 Overview

Complex numbers are an extension of the real number system that introduces a new element called the imaginary unit, denoted as $i$. A complex number $z$ can be expressed in the form $z = a + bi$, where $a$ and $b$ are real numbers. The real part of $z$ is denoted as $\mathrm{Re}(z) = a$, and the imaginary part is denoted as $\mathrm{Im}(z) = b$.




## 2 Operations

Complex numbers support various arithmetic operations:

### 2.1 Addition and Subtraction

To add or subtract complex numbers, simply combine the real and imaginary parts separately. For example, given $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, the sum $z_1 + z_2$ and the difference $z_1 - z_2$ are calculated as follows:

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$$

$$z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i$$

**Example 2.1** ***Addition and Subtraction***

*Consider the complex numbers $z_1 = 3 + 2i$ and $z_2 = -1 + 4i$. We can calculate their sum and difference as follows:*

$$z_1 + z_2 = (3 + 2i) + (-1 + 4i) = 2 + 6i$$

$$z_1 - z_2 = (3 + 2i) - (-1 + 4i) = 4 - 2i$$

## 2.2 Multiplication

To multiply complex numbers, use the distributive property and the fact that $i^2 = -1$. For example, given $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, the product $z_1 \cdot z_2$ is calculated as follows:

$$z_1 \cdot z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = a_1a_2 + a_1b_2 i + b_1a_2 i + b_1b_2 i^2 = (a_1a_2 - b_1b_2) + (a_1b_2 + b_1a_2)i$$

**Example 2.2** ***Multiplication***

*Let's multiply the complex numbers $z_1 = 2 + 3i$ and $z_2 = -4 + i$:*

$$z_1 \cdot z_2 = (2 + 3i)(-4 + i) = -8 + 2i - 12i - 3 = -11 - 10i$$

## 2.3 Division

To divide complex numbers, multiply the numerator and denominator by the conjugate of the denominator. The conjugate of a complex number $z = a + bi$ is denoted as $\bar{z} = a - bi$. For example, given $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, the quotient $\frac{z_1}{z_2}$ is computed as follows:

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i} \cdot \frac{a_2 - b_2 i}{a_2 - b_2 i} = \frac{(a_1a_2 + b_1b_2) + (b_1a_2 - a_1b_2)i}{a_2^2 + b_2^2}$$

**Example 2.3** ***Division***

*Suppose we want to divide $z_1 = 5 + 2i$ by $z_2 = -1 + 3i$. To do this, we multiply both the numerator and denominator by the conjugate of the denominator:*

$$\frac{z_1}{z_2} = \frac{(5 + 2i)(-1 - 3i)}{(-1 + 3i)(-1 - 3i)} = \frac{-5 - 15i - 2i + 6}{1 + 9} = \frac{1 - 17i}{10}$$

# 3 Polar Form

### Conversion from Cartesian to Polar Form:

Given the complex number $z = a + ib$, we can convert it to its polar representation $(r, \theta)$ using the following formulas:

$$r = \sqrt{a^2 + b^2}$$
$$\theta = \arctan\left(\frac{b}{a}\right)$$

Hence, the polar representation of $z$ becomes $(r, \theta)$.

**Conversion from Polar to Cartesian Form:**

Given the polar representation $(r, \theta)$ of a complex number, we can convert it to its Cartesian form $a + ib$ using the following equations:

$$a = r\cos(\theta)$$
$$b = r\sin(\theta)$$

Thus, the Cartesian representation of $(r, \theta)$ becomes $a + ib$.

In addition, the modulus or distance $r$ of the complex number is given by $r = \sqrt{a^2 + b^2}$.

These relationships allow us to convert between the Cartesian and polar forms of complex numbers, expressing them in terms of each other using trigonometric functions like cosine and sine.

**Example 3.1** *Let's consider a complex number $z_1 = 1 + i$. We can convert it to its polar representation $(r_1, \theta_1)$ using the following formulas:*

$$r_1 = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta_1 = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4} \textit{ radians}$$

*Therefore, the polar representation of $z_1$ is $(\sqrt{2}, \frac{\pi}{4})$.*

**Example 3.2** *Let's consider a complex number $z_2 = -2 - 2i$. We can convert it to its polar representation $(r_2, \theta_2)$ using the following formulas:*

$$r_2 = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2}$$

$$\theta_2 = \arctan\left(\frac{-2}{-2}\right) = \arctan(1) = \frac{5\pi}{4} \textit{ radians}$$

*Therefore, the polar representation of $z_2$ is $\left(2\sqrt{2}, \frac{5\pi}{4}\right)$.*

**Example 3.3** *Let's consider a complex number $z_3 = (2, \frac{\pi}{3})$. We can convert it to its Cartesian representation $a_3 + ib_3$ using the following formulas:*

$$a_3 = 2\cos\left(\frac{\pi}{3}\right) = 1$$

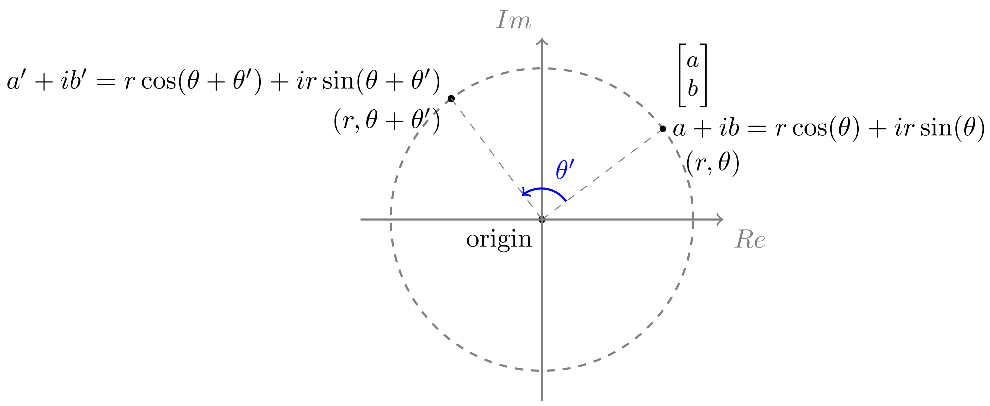
$$b_3 = 2\sin\left(\frac{\pi}{3}\right) = \sqrt{3}$$

*Therefore, the Cartesian representation of $z_3$ is $1 + i\sqrt{3}$.*

# 4 Rotation

Let's consider a point $(r, \theta)$ in the polar form, where $r$ represents the distance from the origin and $\theta$ represents the angle with respect to the positive real axis. Now, we want to rotate this point by an angle $\theta'$:

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} \cos(\theta') & -\sin(\theta') \\ \sin(\theta') & \cos(\theta') \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$




We can achieve the rotation of a point $(r\cos(\theta) + ir\sin(\theta))$ in the complex plane by multiplying it with the complex number $(\cos(\theta') + i\sin(\theta'))$, considering the convention that $i^2 = -1$. The result of this multiplication can be calculated as follows:

$$\begin{aligned}(r\cos(\theta) + ir\sin(\theta))\ (\cos(\theta') + i\sin(\theta')) &= r\cos(\theta)\cos(\theta') + ir\cos(\theta)\sin(\theta') \\ &\quad + ir\sin(\theta)\cos(\theta') + i^2 r\sin(\theta)\sin(\theta') \\ &= r(\cos(\theta)\cos(\theta') - \sin(\theta)\sin(\theta')) \\ &\quad + ir(\cos(\theta)\sin(\theta') + \sin(\theta)\cos(\theta'))\end{aligned}$$

Simplifying further, we can use the angle addition and subtraction formulas for trigonometric functions to obtain:

$$(r\cos(\theta) + ir\sin(\theta))\ (\cos(\theta') + i\sin(\theta')) = r\cos(\theta + \theta') + ir\sin(\theta + \theta')$$

Therefore, the result of this multiplication is the point $(r\cos(\theta + \theta') + ir\sin(\theta + \theta'))$, representing the rotated point in the complex plane.

If we express the point in Cartesian coordinates, we can achieve the rotation by an angle $\theta'$ by multiplying it with the following matrix:

$$\begin{bmatrix} \cos(\theta') & -\sin(\theta') \\ \sin(\theta') & \cos(\theta') \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos(\theta') & -\sin(\theta') \\ \sin(\theta') & \cos(\theta') \end{bmatrix} \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix} = \begin{bmatrix} r\cos(\theta')\cos(\theta) - r\sin(\theta')\sin(\theta) \\ r\sin(\theta')\cos(\theta) + r\cos(\theta')\sin(\theta) \end{bmatrix} = \begin{bmatrix} r\cos(\theta + \theta') \\ r\sin(\theta + \theta') \end{bmatrix}$$

Using the Cartesian notation and the rotation matrix, we can perform rotations in a two-dimensional space without the need for complex numbers.

## 5 Euler's Formula

Euler's formula relates the exponential function, complex numbers, and trigonometric functions. It states that for any real number $\theta$:

$$\mathrm{e}^{\mathrm{i}\theta} = \cos(\theta) + i\sin(\theta)$$

This formula can be proven by expanding the above three functions in power series, using $i^2 = -1$ and grouping real and imaginary terms on the left. The exponential representation makes multiplication and division of complex numbers very easy:

## 5.1 Multiplication

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ be two complex numbers in exponential form. Their product is given by:

$$z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

**Example 5.1** *The product of $z_1 = 2e^{i\frac{\pi}{4}}$ and $z_2 = 3e^{i\frac{\pi}{3}}$ can be calculated as follows:*

$$\begin{aligned} z_1 \cdot z_2 &= (2e^{i\frac{\pi}{4}})(3e^{i\frac{\pi}{3}}) \\ &= 6e^{i\left(\frac{\pi}{4}+\frac{\pi}{3}\right)} \\ &= 6e^{i\frac{7\pi}{12}} \end{aligned}$$

*So, the product of $z_1$ and $z_2$ is $6e^{i\frac{7\pi}{12}}$.*

## 5.2 Division

Similarly, for the division of complex numbers, let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. The quotient is calculated as:

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

**Example 5.2** *Now, let's calculate the quotient $\frac{z_1}{z_2}$:*

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{2e^{i\frac{\pi}{4}}}{3e^{i\frac{\pi}{3}}} \\ &= \frac{2}{3} e^{i\left(\frac{\pi}{4}-\frac{\pi}{3}\right)} \\ &= \frac{2}{3} e^{-i\frac{\pi}{12}} \end{aligned}$$

# 6 Functions of Complex Variables

With the help of power series, many functions that are originally defined for real arguments can be extended to complex numbers. Let's explore a few examples:

**Natural logarithm function:**

The natural logarithm function $\ln(z)$, when $z = re^{i\theta}$, can be expressed as:

$$\ln(z) = \ln(re^{i\theta}) = \ln(r) + \ln(e^{i\theta}) = \ln(r) + i\theta$$

**Example 6.1** *Let's compute the natural logarithm of the imaginary unit $i$ using the given logarithmic identity:*

*Starting with $i = e^{i\frac{\pi}{2}}$ and applying the natural logarithm to both sides, we have:*

$$\ln(i) = \ln\left(e^{i\frac{\pi}{2}}\right) = i\frac{\pi}{2}$$

*Therefore, the natural logarithm of the imaginary unit $i$ is $i\frac{\pi}{2}$.*

**Exponential function:**

The complex exponential on $z = a + ib$ is defined as:

$$e^z = e^{a+ib} = e^a e^{ib} = e^a(\cos(b) + i\sin(b)).$$

# 7 Powers of Complex Numbers

For any complex number $z = re^{i\theta} = r(\cos(\theta) + i\sin(\theta))$ and a positive integer $n$, the $n$th power of $z$ is given by:

$$z^n = r^n e^{in\theta} = r^n(\cos(n\theta) + i\sin(n\theta))$$

This formula allows us to easily calculate the powers of complex numbers by raising the magnitude $r$ to the power $n$ and multiplying the angle $\theta$ by $n$.

**Example 7.1** *Consider the complex number $z = 1+i$. Let's calculate $z^4$. Expanding the binomial, we get*

$$(1+i)^4 = 1 + 4i + 6i^2 + 4i^3 + i^4.$$

*Simplifying, we have*

$$(1+i)^4 = 1 + 4i - 6 - 4i + 1 = -4 + 0i = -4.$$

*Expressing $z = 1 + i$ in exponential form, we have $z = \sqrt{2}e^{i\frac{\pi}{4}}$. To find $z^4$, we can raise the magnitude to the power of 4 and multiply the angle by 4.*

$$z^4 = (\sqrt{2})^4 e^{i4\frac{\pi}{4}} = 4(\cos(\pi) + i\sin(\pi)) = -4$$

# 8 Roots of Complex Numbers

Finding the $n$th root of a complex number is the inverse operation of raising a complex number to a power. The $n$th root of a complex number $z$ is given by:

$$z^{1/n} = \left\{ \sqrt[n]{r}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right) | k = 0, 1, 2, ..., n-1 \right\}$$

This formula allows us to find all the $n$th roots of a complex number by taking the $n$th root of the magnitude $r$ and dividing the angle $\theta$ by $n$.

The number of distinct $n$th roots of a complex number is $n$. Each root corresponds to a different value of $k$ in the formula, ranging from 0 to $n - 1$.

**Example 8.1** *Consider the complex number $w = 2\sqrt{3} + 2i$. Let's find the cube roots of $w$.*

*To calculate the cube roots of $w$, we can use the formula $w^{1/3} = \sqrt[3]{|w|}\left(\cos\left(\frac{\theta+2k\pi}{3}\right) + i\sin\left(\frac{\theta+2k\pi}{3}\right)\right)$, where $k = 0, 1, 2$ represents the three distinct roots.*

*First, we need to find the magnitude and the angle of $w$. The magnitude $|w|$ can be calculated as $\sqrt{(2\sqrt{3})^2 + 2^2} = 4$, and the angle $\theta$ can be determined using $\theta = \arctan\left(\frac{2}{2\sqrt{3}}\right) = \frac{\pi}{6}$.*

*Substituting these values into the formula, we can find the cube roots of $w$:*

$$
\begin{aligned}
w_0^{1/3} &= \sqrt[3]{4}\left(\cos\left(\frac{\frac{\pi}{6}+2\cdot 0\pi}{3}\right)+i\sin\left(\frac{\frac{\pi}{6}+2\cdot 0\pi}{3}\right)\right)\\
w_1^{1/3} &= \sqrt[3]{4}\left(\cos\left(\frac{\frac{\pi}{6}+2\cdot 1\pi}{3}\right)+i\sin\left(\frac{\frac{\pi}{6}+2\cdot 1\pi}{3}\right)\right)\\
w_2^{1/3} &= \sqrt[3]{4}\left(\cos\left(\frac{\frac{\pi}{6}+2\cdot 2\pi}{3}\right)+i\sin\left(\frac{\frac{\pi}{6}+2\cdot 2\pi}{3}\right)\right)
\end{aligned}
$$

*Simplifying further, we find:*

$$
\begin{aligned}
w_0^{1/3} &= \sqrt[3]{4}\left(\cos\left(\frac{\pi}{18}\right)+i\sin\left(\frac{\pi}{18}\right)\right)\\
w_1^{1/3} &= \sqrt[3]{4}\left(\cos\left(\frac{13\pi}{18}\right)+i\sin\left(\frac{13\pi}{18}\right)\right)\\
w_2^{1/3} &= \sqrt[3]{4}\left(\cos\left(\frac{25\pi}{18}\right)+i\sin\left(\frac{25\pi}{18}\right)\right).
\end{aligned}
$$

**Example 8.2** *The equation $x^4+1=0$ does not have any real roots. The equation can be rearranged as $x^4=-1$. We know that the square of any real number is always non-negative. Since the right-hand side of the equation is $-1$, which is negative, it implies that there are no real numbers whose square is equal to $-1$. Therefore, the equation $x^4+1=0$ has no real roots. However, it does have complex roots. Using exponential form $-1=e^{i\pi}$, now we can express $x=\sqrt[4]{-1}$ as :*

$$x=\{e^{i\left(\frac{\pi+2k\pi}{4}\right)}\}$$

*where $k$ ranges from $0$ to $3$ to obtain the four distinct roots. Then, we have*

$$x=\{e^{i\left(\frac{\pi}{4}\right)},e^{i\left(\frac{3\pi}{4}\right)},e^{i\left(\frac{5\pi}{4}\right)},e^{i\left(\frac{7\pi}{4}\right)}\}.$$