Quadratic forms

Wednesday 23.08.2023 (Room M1)

1 Introduction

In its simplest form, a quadratic form Q in n variables $x_1, x_2, ..., x_n$ is expressed as:

$$Q(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

where a_{ij} are real or complex numbers, and these numbers are symmetric, meaning $a_{ij} = a_{ji}$.

2 Matrix Representation

Quadratic forms are closely tied to matrices. Given a symmetric matrix A of size $n \times n$ with entries a_{ij}, a quadratic form Q is represented as:

$$Q(x) = x^T A x$$

This matrix representation simplifies many operations on Q.

Example 2.1 *Consider the following quadratic form:*

$$Q(x, y, z) = x^2 + y^2 + z^2 + 4xy - 2xz + 6yz$$

The given quadratic form can be represented by the symmetric matrix A:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

Given the vector

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

the quadratic form is

$$Q(x, y, z) = \mathbf{x}^T A \mathbf{x} = \begin{bmatrix} x, y, z \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

3 Principal Axes Theorem

Every real symmetric matrix can be orthogonally diagonalized. This implies that for any symmetric matrix A, there exists an orthogonal matrix P (i.e., $P^{-1} = P^T$) such that:

$$A = PDP^T$$

Where D is a diagonal matrix. This diagonalization allows for a change of variable to express Q in a simplified form without cross terms.

For example a quadratic form $Q(x, y, z)$ can be transformed using the orthogonal transformation into a simpler form:

$$Q'(x', y', z') = \lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2$$

Here, x', y', and z' are coordinates in the new orthogonal system defined by the eigenvectors of A.

Example 3.1 *Consider the quadratic form:*

$$Q(x, y) = 3x^2 + 3y^2 + 4xy$$

We aim to express this form in terms of the principal axes using the Principal Axes Theorem. Firstly, the quadratic form can be represented using the symmetric matrix A:

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

Given a vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, *the quadratic form is expressed as:*

$$Q(x, y) = \mathbf{x}^T A \mathbf{x}$$

To determine the eigenvalues of A, we solve the characteristic equation:

$$det(A - \lambda I) = 0$$

This yields:

$$\lambda^2 - 6\lambda + 5 = 0$$

The eigenvalues are:

$$\lambda_1 = 1, \quad \lambda_2 = 5$$

To determine the eigenvectors of A, we solve the equation $(A - \lambda_i I)\mathbf{v} = 0$:

For λ_1: Solving $(A - \lambda_1 I)\mathbf{v}_1 = 0$, we get:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \implies \mathbf{u}_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

For λ_2: Solving $(A - \lambda_2 I)\mathbf{v}_2 = 0$, we get:

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies \mathbf{u}_2 = \frac{\mathbf{v}_2}{|\mathbf{v}_2|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Forming the orthogonal matrix P from the eigenvectors and D from the eigenvalues :

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\begin{aligned} Q(x, y) &= 3x^2 + 3y^2 + 4xy \\ &= \begin{bmatrix} x, y \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} x, y \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} -x/\sqrt{2} + y/\sqrt{2}, x/\sqrt{2} + y/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -x/\sqrt{2} + y/\sqrt{2} \\ x/\sqrt{2} + y/\sqrt{2} \end{bmatrix} \\ &= (-x/\sqrt{2} + y/\sqrt{2})^2 + 5(x/\sqrt{2} + y/\sqrt{2})^2 \end{aligned}$$

The quadratic form $Q(x, y)$ has been transformed to:

$$Q'(x', y') = x'^2 + 5y'^2$$

where $x' = -x/\sqrt{2} + y/\sqrt{2}$ and $y' = x/\sqrt{2} + y/\sqrt{2}$ are coordinates in the new orthogonal system defined by the eigenvectors of A.

Example 3.2 *Consider the quadratic form:*

$$Q(x, y, z) = 2xy + 2xz + 2yz$$

Our objective is to express this quadratic form in terms of the principal axes using the Principal Axes Theorem.

The quadratic form can be represented as:

$$Q(x, y, z) = \mathbf{x}^T A \mathbf{x}$$

where $\mathbf{x}$ is the column vector

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and A is the symmetric matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

From previous computations, we found an orthonormal matrix P and a diagonal matrix D such that $A = PDP^T$:

$$P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

To transform to the new orthogonal coordinate system (x', y', z'), we apply:

$$\left[x', y', z'\right]^T = P^T \left[x, y, z\right]^T$$

Thus, we have:

$$\begin{aligned} x' &= -x/\sqrt{2} + y/\sqrt{2} \\ y' &= -x/\sqrt{6} - y/\sqrt{6} + 2z/\sqrt{6} \\ z' &= x/\sqrt{3} + y/\sqrt{3} + z/\sqrt{3} \end{aligned}$$

In terms of the new coordinates, the quadratic form becomes:

$$Q'(x', y', z') = -x'^2 - y'^2 + 2z'^2.$$

4 Classification of Quadratic Forms

Classifying quadratic forms based on the eigenvalues of their associated matrices provides a structured way to understand their behavior. It gives insights into optimization problems, stability analyses, and many other applications where the nature of the curvature or shape of the graph matters.

The nature of the graph of $Q(x)$ - whether it represents valleys, peaks, or saddles in multi-dimensional space - is determined by the eigenvalues of A. Let's delve deeper into each classification:

4.1 Positive definite

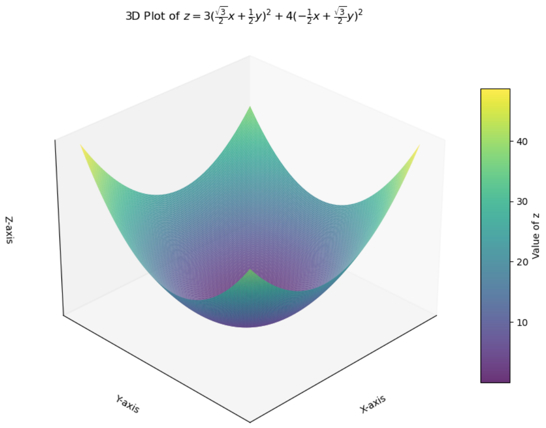
A matrix A is said to be positive definite if all its eigenvalues are positive. This implies that the quadratic form $Q(x)$ is always positive for all non-zero vectors x. Graphically, this represents a surface that opens upwards, resembling a bowl.

$$Q(x) > 0 \, \text{for all} \, x \neq 0$$

Example 4.1 *Consider the quadratic form*

$$\begin{aligned}
Q_1 &= \frac{13x^2}{4} - \frac{\sqrt{3}xy}{2} + \frac{15y^2}{4} \\
&= [x, y] \begin{bmatrix} \frac{13}{4} & \frac{-\sqrt{3}}{4} \\ \frac{-\sqrt{3}}{4} & \frac{15}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= [x, y] \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= 3(\frac{\sqrt{3}}{2}x + \frac{1}{2}y)^2 + 4(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y)^2
\end{aligned}$$

Below is a graphical representation of Q_1*:*



4.2 Negative definite

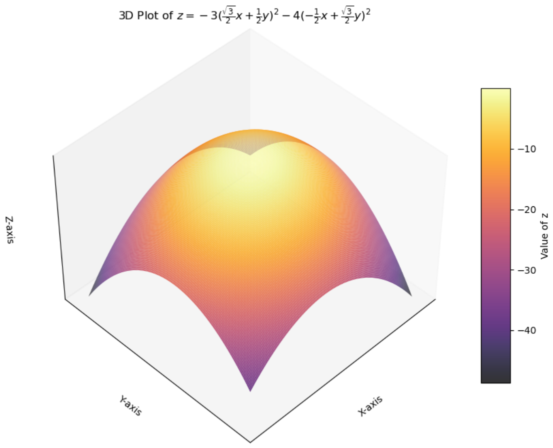
Conversely, a matrix that has all negative eigenvalues defines a negative definite quadratic form. For such matrices, $Q(x)$ is always negative for all non-zero vectors x. The graph of this form is a surface opening downwards, like an upside-down bowl.

$$Q(x) < 0 \, \text{for all} \, x \neq 0$$

Example 4.2 *Consider the quadratic form*

$$\begin{aligned} Q_2 &= -\frac{13x^2}{4} + \frac{\sqrt{3}xy}{2} - \frac{15y^2}{4} \\ &= [x, y] \begin{bmatrix} \frac{-13}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{-15}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= [x, y] \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= -3(\frac{\sqrt{3}}{2}x + \frac{1}{2}y)^2 - 4(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y)^2 \end{aligned}$$

Below is a graphical representation of Q_2*:*



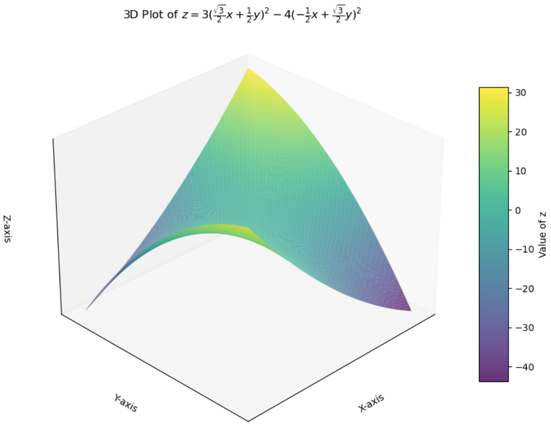
4.3 Indefinite

Quadratic forms associated with matrices that have both positive and negative eigenvalues are termed as indefinite. These forms do not consistently stay above or below the x-axis, leading to saddle points.

Example 4.3 *Consider the quadratic form*

$$\begin{aligned} Q_3 &= \frac{5x^2}{4} + \frac{7\sqrt{3}xy}{2} - \frac{9y^2}{4} \\ &= [x, y] \begin{bmatrix} \frac{5}{4} & \frac{7\sqrt{3}}{4} \\ \frac{7\sqrt{3}}{4} & \frac{9}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= [x, y] \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= 3(\frac{\sqrt{3}}{2}x + \frac{1}{2}y)^2 - 4(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y)^2 \end{aligned}$$

Below is a graphical representation of Q_3*:*



4.4 Positive semi-definite

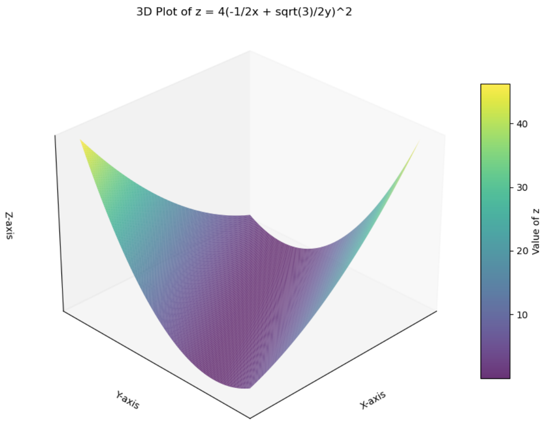
For a matrix to be positive semi-definite, all its eigenvalues must be non-negative. This suggests that $Q(x)$ is either positive or zero for all vectors x. It's like a bowl that has been flattened at some regions.

$$Q(x) \geq 0 \,\text{for all}\, x$$

Example 4.4 *Consider the quadratic form*

$$\begin{aligned} Q_4 &= x^2 - 2\sqrt{3}xy + 3y^2 \\ &= [x, y] \begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= [x, y] \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= 4(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y)^2 \end{aligned}$$

Below is a graphical representation of Q_4:



4.5 Negative semi-definite

Similarly, a matrix with all non-positive eigenvalues leads to a negative semi-definite form. This means that $Q(x)$ is either negative or zero for all vectors x. It represents an inverted bowl with some flat regions.

$$Q(x) \leq 0 \text{ for all } x$$

Example 4.5 *Consider the quadratic form*

$$\begin{aligned} Q_5 &= -x^2 + 2\sqrt{3}xy - 3y^2 \\ &= [x, y] \begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= [x, y] \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= -4(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y)^2 \end{aligned}$$

Below is a graphical representation of Q_5*:*