Eigenvalues and Eigenvectors

Wednesday 16.08.2023 (Room M1)

1 Overview

Introduction to Eigenvalues and Eigenvectors

In the vast domain of linear algebra, matrices represent linear transformations — rules by which vectors are stretched, shrunk, rotated, or otherwise changed. But amid these transformations, some vectors have a unique property: when acted upon by a matrix, they only stretch or shrink, without changing their direction. These special vectors are called "eigenvectors," and the factor by which they stretch or shrink is termed as their "eigenvalue."

Definition: Let A be a square matrix. A vector v is said to be an *eigenvector* of A if, and only if, there exists a scalar λ, referred to as an *eigenvalue*, associated with v such that:

$$Av = \lambda v \tag{1}$$

Eigenvectors are special vectors associated with matrix A that remain unchanged in direction after the matrix's transformation. The eigenvalue, λ, indicates the factor by which the eigenvector is stretched or shrunk during the transformation. If λ is negative, the eigenvector is also reflected. Together, the eigenvectors and eigenvalues of a matrix provide a profound understanding of the matrix's geometric transformation.

1.1 Determining Eigenvalues

Given a matrix A, λ is an eigenvalue of A if there exists a non-zero vector v such that $Av = \lambda v$. The vector v is called an eigenvector associated with λ. Starting from the equation $Av = \lambda v$, we can rewrite it as:

$$Av - \lambda v = 0$$

Which is equivalent to:

$$(A - \lambda I)v = 0$$

Where I is the identity matrix of the same size as A.

For this equation to have a non-trivial (non-zero) solution for v, the matrix $A - \lambda I$ must be noninvertible (singular). This implies that:

$$\det(A - \lambda I) = 0$$

This equation gives us the characteristic polynomial of matrix A. The values of λ that satisfy this equation are the eigenvalues of A. This determinant is a polynomial with the variable of λ, that is called the characteristic polynomial.

Mathematically expressed:

$$P(\lambda) = \det(A - \lambda I)$$

Thus the eigenvalues of A are the values of λ for which this polynomial is zero. The determinant condition, $\det(A - \lambda I) = 0$, is fundamental in determining the eigenvalues of a matrix. It's derived from the requirement that there exist non-trivial vectors v which satisfy the eigenvalue equation, ensuring the matrix transformation only scales and does not rotate or shear the vector.

Example 1.1 *Consider the matrix:*

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$$

The characteristic polynomial is:

$$P(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix}$$

Expanding this:

$$P(\lambda) = (3-\lambda)(-\lambda) - 2(-1) \quad \Rightarrow \quad \lambda^2 - 3\lambda + 2 = 0$$

Solving this quadratic equation, the eigenvalues are $\lambda_1 = 2$ *and* $\lambda_2 = 1$.

Example 1.2 *Consider the matrix:*

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

Here, the matrix is upper triangular, so the eigenvalues are simply the diagonal elements. However, for the sake of demonstration, let's compute the characteristic polynomial:

$$P(\lambda) = \det(B - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 5-\lambda \end{vmatrix}$$

Expanding this:

$$P(\lambda) = (2-\lambda)(3-\lambda)(5-\lambda) \quad \Rightarrow \quad \lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$$

The eigenvalues of B *are thus* $\lambda_1 = 2$, $\lambda_2 = 3$, *and* $\lambda_3 = 5$.

For matrices larger than 3×3, *analytical solutions can become intricate. In such cases, numerical methods or software tools are often employed to compute the eigenvalues.*

2 Properties of Eigenvalues

We'll delve into some of these properties, providing deeper insight into their significance.

1. Eigenvalues of Diagonal Matrices

A diagonal matrix is characterized by having non-zero values only along its main diagonal. For such a matrix

$$D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix}$$

the eigenvalues are directly the diagonal elements, i.e., $d_1, d_2, \dots, d_n$. This is because the diagonal form itself represents a scaling transformation.

2.1 Complexity

Eigenvalues can be either real or complex. Intriguingly, matrices with solely real entries can yield complex eigenvalues. As an illustration:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

This matrix has the eigenvalues i and $-i$, which are, notably, complex.

Example 2.1 *Consider a matrix A representing a rotation in the $2D$ plane:*

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

where θ denotes the angle of rotation in a counter-clockwise direction.

When the matrix acts on a real vector v in the plane, it rotates the vector by an angle θ but doesn't change its length. This means the rotated vector won't lie on its original path anymore, except for the zero vector $\mathbf{0}$. Simply, the matrix doesn't just stretch or shrink vectors—it spins them! So, while real vectors don't fit the standard eigenvalue equation $Av = \lambda v$ with a real λ, when we consider complex eigenvalues, we can find fitting eigenvectors and eigenvalues.

To determine the eigenvalues, we use the characteristic polynomial:

$$\det(A - \lambda I) = 0$$

This leads to:

$$\det \begin{bmatrix} \cos(\theta) - \lambda & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) - \lambda \end{bmatrix} = 0$$

The solutions to the ensuing equation:

$$\lambda^2 - 2\lambda\cos(\theta) + 1 = 0$$

yield the eigenvalues:

$$\lambda_1 = e^{i\theta}$$
$$\lambda_2 = e^{-i\theta}$$

Unless $\theta = 0$ or $\theta = \pi$ (where they become real), these eigenvalues are complex.

2.2 Spectral Theorem

A symmetric matrix, identifiable by its equality to its transpose, will always have real eigenvalues. However, having real eigenvalues isn't exclusive to symmetric matrices; non-symmetric matrices can still possess entirely real eigenvalues.

Example 2.2 *Given the symmetric matrix:*

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

Subtracting λ from the diagonal entries of A, we get:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{bmatrix}$$

Now, we compute the determinant of $A - \lambda I$:

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{bmatrix}$$

Expanding along the top row:

$$\det(A - \lambda I) = (1-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ 0 & 1-\lambda \end{vmatrix} + 0 \begin{vmatrix} 2 & 1-\lambda \\ 0 & 2 \end{vmatrix}$$

Computing each of the 2×2 determinants :

For the first minor:

$$(1-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\left((1-\lambda)(1-\lambda) - 4\right) = (1-\lambda)(\lambda^2 - 2\lambda - 3)$$

For the second minor:

$$-2 \begin{vmatrix} 2 & 2 \\ 0 & 1-\lambda \end{vmatrix} = -2(2(1-\lambda) - 0) = -4(1-\lambda)$$

Now, combining:

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda)(\lambda^2 - 2\lambda - 3) - 4(1-\lambda) \\ &= (1-\lambda)((\lambda^2 - 2\lambda - 3) - 4) = (1-\lambda)(\lambda^2 - 2\lambda - 7) \end{aligned}$$

This polynomial is our characteristic equation. Solving this equation yields the eigenvalues. For the equation $\lambda^2 - 2\lambda - 7 = 0$, the solutions can be found using the quadratic formula:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where:

$$a = 1, \quad b = -2, \quad c = -7$$

The eigenvalues are:

$$\lambda_1 = 1, \quad \lambda_2 = 1 - 2\sqrt{2}, \quad \lambda_3 = 1 + 2\sqrt{2}$$

Example 2.3 *Given the non-symmetric matrix:*

$$B = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

To find the eigenvalues for the matrix, we set up and solve the characteristic equation:

$$\det(B - \lambda I) = 0$$

This yields:

$$\det(B - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 1 \\ 1 & 3-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{bmatrix} = 0$$

Expanding along the top row:

$$\det(B - \lambda I) = (2-\lambda) \begin{vmatrix} 3-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} - 0 + 1 \begin{vmatrix} 1 & 3-\lambda \\ 1 & 1 \end{vmatrix}$$

Computing the determinants:

$$\begin{aligned} \det(B - \lambda I) &= (2-\lambda)(3-\lambda)(2-\lambda) + 1 - (3-\lambda) \\ &= (2-\lambda)(3-\lambda)(2-\lambda) - (2-\lambda) \\ &= (2-\lambda)((3-\lambda)(2-\lambda) - 1) \\ &= (2-\lambda)(\lambda^2 - 5\lambda + 5) \end{aligned}$$

This polynomial is our characteristic equation. Solving this equation yields the eigenvalues. For the equation $\lambda^2 - 5\lambda + 5 = 0$, the solutions can be found using the quadratic formula:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where:

$$a = 1, \quad b = -5, \quad c = 5$$

The eigenvalues are:

$$\lambda_1 = 2, \quad \lambda_2 = \frac{5 - \sqrt{5}}{2}, \quad \lambda_3 = \frac{5 + \sqrt{5}}{2}$$

Example 2.4 *Given another non-symmetric matrix:*

$$C = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

To find the eigenvalues, we compute the determinant of $C - \lambda I$, where λ are the eigenvalues and I is the identity matrix:

$$C - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{bmatrix}$$

The characteristic polynomial is:

$$\det(C - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{vmatrix}$$

Expanding:

$$\det(C - \lambda I) = -\lambda(\lambda^2 + 1) - \lambda$$

Simplifying, we obtain:

$$-\lambda^3 - 2\lambda = 0$$

Factoring:

$$-\lambda(\lambda^2 + 2) = 0$$

Solving $\lambda^2 + 2$ gives:

$$\lambda_{2,3} = \pm\sqrt{2}\,i$$

Thus, the eigenvalues are:

$$\lambda_1 = 0, \quad \lambda_2 = -\sqrt{2}\,i, \quad \lambda_3 = \sqrt{2}\,i$$

2.3 Trace and Determinant in terms of Eigenvalues

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of a matrix A, then:

1. **Trace:** The trace of A (sum of its diagonal elements) is the sum of its eigenvalues:

$$\text{trace}(A) = \sum_{i=1}^{n} \lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

2. **Determinant:** The determinant of A is the product of its eigenvalues:

$$\det(A) = \prod_{i=1}^{n} \lambda_i = \lambda_1 \times \lambda_2 \times \cdots \times \lambda_n$$

Example 2.5 *For the matrix B in example 2.3*

$$B = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

and its eigenvalues:

$$\lambda_1 = 2, \quad \lambda_2 = \frac{5-\sqrt{5}}{2}, \quad \lambda_3 = \frac{5+\sqrt{5}}{2}$$

We can verify the properties associated with the trace and determinant of matrix B:

1. Trace:

$$\begin{aligned} &trace(B) = 2 + 3 + 2 = 7 \\ &\lambda_1 + \lambda_2 + \lambda_3 = 2 + \frac{5-\sqrt{5}}{2} + \frac{5+\sqrt{5}}{2} = 7 \end{aligned}$$

Thus, the sum of the eigenvalues matches the trace of the matrix, which is $trace(B) = 7$.

2. Determinant:

$$\begin{aligned} &\det(B) = 2(6-0) + 1(1-3) = 10 \\ &\lambda_1 \times \lambda_2 \times \lambda_3 = 2 \times (\frac{5-\sqrt{5}}{2}) \times (\frac{5+\sqrt{5}}{2}) = 10 \end{aligned}$$

Thus, the product of the eigenvalues matches the determinant of the matrix, which is $\det(B) = 10$.

Example 2.6 *For the matrix C in example 2.4*

$$C = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

and its eigenvalues:

$$\lambda_1 = 0, \quad \lambda_2 = -\sqrt{2}\,i, \quad \lambda_3 = \sqrt{2}\,i,$$

We can verify the properties associated with the trace and determinant of matrix C:

1. Trace:

$$\begin{aligned} &trace(C) = 0 \\ &\lambda_1 + \lambda_2 + \lambda_3 = 0 - \sqrt{2}\,i + \sqrt{2}\,i = 0 \end{aligned}$$

Thus, the sum of the eigenvalues matches the trace of the matrix, which is $trace(C) = 0$.

2. Determinant:

$$\begin{aligned} &\det(C) = 0 \\ &\lambda_1 \times \lambda_2 \times \lambda_3 = 0 \times (-\sqrt{2}\,i) \times \sqrt{2}\,i = 0 \end{aligned}$$

Thus, the product of the eigenvalues matches the determinant of the matrix, which is $\det(C) = 0$.

2.4 Relationship between Eigenvalues and the Inverse of a Matrix

A matrix is noninvertible (singular) if and only if it has an eigenvalue of 0. This is because the determinant of the matrix is zero, and as mentioned before, the product of eigenvalues is equal to the determinant. Thus the matrix C in example 2.4 is noninvertible, because it has a zero eigenvalue. But the matrix B in example 2.3 is invertible since doesn't have any zero eigenvalues.

If a matrix A has an eigenvalue λ, and assuming A is invertible (its determinant is non-zero), then its inverse A^{-1} will have an eigenvalue of $\frac{1}{\lambda}$. This makes intuitive sense since the inverse matrix undoes the transformation of the original matrix.

Example 2.7 *Consider the following* 3×3 *matrix* A*:*

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To find the eigenvalues of A*, we compute the characteristic polynomial:*

$$\det(A - \lambda I) = (1 - \lambda)(\lambda^2 + 1) = 0$$

from which we have:

$$\lambda = \pm i \qquad \textit{or} \qquad \lambda = 1$$

The eigenvalues of A *are thus* $\lambda_1 = i$*,* $\lambda_2 = -i$*, and* $\lambda_3 = 1$*.*

The inverse of a matrix will have the reciprocal of its eigenvalues, so the eigenvalues of A^{-1} *will be* $1/i = -i$*,* $1/(-i) = i$*, and* $1/1 = 1$*.*

2.5 Similar Matrices have the Same Eigenvalues

Matrix similarity is a transformation-preserving relation between two matrices. Formally, two matrices A and B are said to be similar if there exists an invertible matrix X such that:

$$A = X^{-1}AX.$$

If matrices A and B are similar, then they have the same eigenvalues.

Proof: Assume matrices A and B are similar. By definition, there exists an invertible matrix P such that

$$A = X^{-1}BX$$

Consider the characteristic polynomial of A:

$$\begin{aligned} \det(\lambda I - A) &= \det(\lambda I - X^{-1}BX) \\ &= \det(\lambda X^{-1}IX - X^{-1}BX) \\ &= \det(X^{-1}(\lambda I - B)X) \\ &= \det(X^{-1})\det(\lambda I - B)\det(X) \\ &= \frac{1}{\det(X)}\det(\lambda I - B)\det(X) \\ &= \det(\lambda I - B) \end{aligned}$$

This is the characteristic polynomial of B, so A and B have the same characteristic polynomial. Hence A and B have the same eigenvalues.

2.6 Transpose and Eigenvalues

The $n \times n$ matrix A and its transpose A^T have the same eigenvalues.

Proof: The characteristic polynomial for A^T is:

$$|A^T - \lambda I|$$

Since λI has only nonzeros on main diagonal, we can see that $A^T - \lambda I = (A - \lambda I)^T$

Using the property that the determinant of a matrix and its transpose are identical, we rewrite this as:

$$|A^T - \lambda I| = |(\lambda I - A)^T| = |\lambda I - A|$$

This is the characteristic polynomial of A. Thus, A^T and A have the identical characteristic polynomial. As the eigenvalues are the roots of the characteristic polynomial, A and A^T share the same eigenvalues.

2.7 Finding Eigenvectors

Given a matrix A, once its eigenvalues λ are determined, we proceed to compute the associated eigenvectors. The eigenvectors corresponding to the eigenvalue λ lie in the eigenspace defined by the null space of $(A - \lambda I)$, where I denotes the identity matrix of the same dimension as A. This relationship can be represented by the homogeneous system:

$$(A - \lambda I)x = 0. \tag{2}$$

A crucial property of eigenvectors is their indeterminacy up to a scalar multiple. That is, if v is an eigenvector of A associated with the eigenvalue λ, then every scalar multiple of v remains an eigenvector of A corresponding to λ. This can be illustrated as:

$$Av = \lambda v. \tag{3}$$

To further elaborate on this concept, consider a scalar c and the vector cv. By multiplying this vector by A, we deduce:

$$\begin{aligned} A(cv) &= c(Av) \\ &= c(\lambda v) \\ &= \lambda(cv), \end{aligned}$$

which substantiates that cv is indeed an eigenvector of A associated with the eigenvalue λ. Thus, the eigenspace or null space of $(A - \lambda I)$ encapsulates all eigenvectors corresponding to λ, each only differing by a scalar multiple.

Example 2.8 *Consider the matrix*

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix},$$

which possesses the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$.

Eigenvalue $\lambda_1 = 3$*: To determine the associated eigenvectors, we compute:*

$$(A - 3I)v = 0,$$

where $A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$. *Solving the linear system*

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

we find that $y = 0$ *with* x *being arbitrary. Thus, the eigenspace for* λ_1 *is given by*

$$N(A - 3I) = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in \mathbb{R} \right\} = \left\{ x \begin{bmatrix} 1 \\ 0 \end{bmatrix} \,\middle|\, x \in \mathbb{R} \right\},$$

yielding the representative eigenvector $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Eigenvalue $\lambda_2 = 2$*: Similarly, we compute:*

$$(A - 2I)v = 0,$$

where $A - 2I = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. *For the system*

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

we deduce $x = -y$. *The eigenspace for* λ_2 *is then*

$$N(A - 2I) = \left\{ \begin{bmatrix} -y \\ y \end{bmatrix} \mid y \in \mathbb{R} \right\} = \left\{ y \begin{bmatrix} -1 \\ 1 \end{bmatrix} \,\middle|\, y \in \mathbb{R} \right\},$$

yielding the representative eigenvector $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.