Null Spaces

Wednesday 09.08.2023 (Room M1)

1 Bases of Cartesian Product $\mathbb{R}^n$

Based on definition, a basis for $\mathbb{R}^n$ is a set of vectors that satisfies two conditions:

- The vectors span $\mathbb{R}^n$, meaning that any vector in $\mathbb{R}^n$ can be expressed as a linear combination of the basis vectors.
- The vectors are linearly independent, implying that no vector in the basis can be written as a linear combination of the others.

In the context of linear independence, removing any vector from the basis would reduce its ability to span the entire space. This is why the basis vectors are sometimes likened to a "building kit" for the vector space: Just like how you can't build a specific model without all the necessary building blocks, you can't cover the entire vector space without all the basis vectors.

Therefore, a basis is indeed the "minimal" set of vectors that can describe the entire space. This principle applies not only to finite dimensional spaces like $\mathbb{R}^n$, but to any vector space in general.

Example 1.1 ***Basis in*** $\mathbb{R}^3$

Consider the following set, which consists of three vectors in $\mathbb{R}^3$*:*

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

To show that these vectors form a basis for $\mathbb{R}^3$*, we need to demonstrate two things: span and linear independence.*

Span: *It means that any vector* $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *in* $\mathbb{R}^3$ *can be written as a combination of these vectors:*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This shows that the vectors span $\mathbb{R}^3$ *since we can represent any vector in* $\mathbb{R}^3$ *using these vectors.*

Linear Independence: *It means that the only way to satisfy the equation*

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is by setting $c_1 = c_2 = c_3 = 0$*. This implies that the coefficients of the vectors must all be zero for the equation to hold.*

By solving the system of equations:

$$\begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$$

we find that the only solution is $c_1 = c_2 = c_3 = 0$. Hence, these vectors are linearly independent. The vectors form a basis for the vector space $\mathbb{R}^3$. This particular set of vectors is referred to as the 'standard basis' for $\mathbb{R}^3$.

Example 1.2 *The set of vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ forms a basis for $\mathbb{R}^3$, we need to show that they are linearly independent and span $\mathbb{R}^3$. A matrix A that its rows are $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ is as follows:*

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus the LU factorization of A we have $L = I$ is identity matrix and $U = A$. Because A is already in row-echelon form and we don't need to do any elemnntary row operations. We also know that rows in U are linearly independent. Therefore rows of A are linearly independent.

To show that the vectors span $\mathbb{R}^3$, we need to show that any vector $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ can be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Thus we need to find c_1, c_2, and c_3 such that:

$$\begin{bmatrix} c_1 \\ c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \\ c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_1 + c_2 \\ c_1 + c_2 + c_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This is achieved by setting $c_1 = x$, $c_2 = y - x$, and $c_3 = z - y$. Since the vectors are linearly independent and span $\mathbb{R}^3$, they form a basis for $\mathbb{R}^3$.

From what we've seen in examples 1.1 and 1.2, it becomes evident that a vector space can possess more than just one basis. However, regardless of the specific basis chosen, the number of vectors that make up these bases remains constant. This invariant quantity corresponds to the maximum number of linearly independent vectors that can exist in the given space.

2 Rank and Bases

The rank of a space, often referred to as its dimension, is the number of vectors in a basis, or equivalently, the maximum number of linearly independent vectors it can accommodate.

In the space $\mathbb{R}^n$, the number of vectors forming a basis is precisely n. These vectors are not only linearly independent but also sufficiently representative to span the entirety of the $\mathbb{R}^n$ space. In other words, given n maximally linearly independent vectors (let's denote them as $v_1, v_2, \ldots, v_n$), any other vector v within the space can be expressed as a linear combination of these basis vectors. If this were not true, it would contradict the premise that n is the maximum number of linearly independent vectors.

A basis for $\mathbb{R}^n$ can be represented by an n by n square matrix, where each vector from the basis is treated as a row in the matrix. We can see this representation in the following $\mathbb{R}^4$ bases:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In the reverse scenario, if we have a square matrix of dimensions n by n and the rows are linearly independent, these rows can form a basis for $\mathbb{R}^n$. This is because they are linearly independent and they form a set of maximum size that can span the space $\mathbb{R}^n$. Hence, any vector in $\mathbb{R}^n$ can be written as a linear combination of these n rows.

This brings us to the concept of the rank of a matrix. The rank of an n by n matrix is the maximum number of linearly independent rows (or columns) in the matrix. If the rows of a square matrix are linearly independent, then they form a basis for $\mathbb{R}^n$. Any vector in $\mathbb{R}^n$ can be written as a linear combination of these n rows, demonstrating that they span $\mathbb{R}^n$.

This correlation between bases in linear algebra and square matrices is quite significant. It paves the way for numerous applications of linear algebra, such as solving linear equation systems, effecting transformations in computer graphics, and even facilitating advancements in quantum mechanics.

Example 2.1 *Consider the* 3 *by* 3 *matrix:*

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

This matrix has rank 3 *because all its rows (and columns) are linearly independent. Since this is a* 3 *by* 3 *matrix, its rows can form a basis for* $\mathbb{R}^3$.

Example 2.2 *Consider a* 2 *by* 2 *matrix:*

$$\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$$

This matrix also has rank 2*, as both rows (and columns) are linearly independent. None of them can be expressed as a linear combination of the other. Hence, the rows of this matrix form a basis for* $\mathbb{R}^2$.

3 Rank of a Matrix and Linearly Independent Rows and Columns

In the study of linear algebra, the rank of a matrix is a crucial concept, closely tied to the linear independence of its rows and columns.

An important property of any matrix is that the number of linearly independent rows is always equal to the number of linearly independent columns. This is because any operation that you can perform on the rows of a matrix can also be performed on the columns, and vice versa.

The rank of a matrix, denoted as $rank(A)$ for a matrix A, is defined as the maximum number of linearly independent rows or columns. It essentially measures the dimension of the row space or the column space of the matrix.

Furthermore, the rank of a matrix has a significant relationship with its invertibility. Specifically, a square matrix A of size n by n is invertible if and only if its rank is n.

If every entry on the main diagonal of an upper (or lower) triangular matrix is non-zero, then the rows of the matrix are linearly independent. Thus a square, upper (or lower) triangular matrix, where all the diagonal entries are non-zero, is invertible.

Example 3.1 *Let's consider a* 3×3 *matrix A:*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The entry on main diagonal of A are nonzero thus A is invertible. We can also calculate the inverse of A. We augment the given matrix A with the identity matrix and perform Gauss-Jordan elimination:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

In the first step, we subtract 2 *times the second row from the first row. Then, in the second step, we subtract the third row from the first and second row :*

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Since we have reduced the left side to the identity matrix, the right side now gives us A^{-1}*:*

$$A^{-1} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

We see that all three rows are linearly independent. Similarly, all three columns are also linearly independent. Hence, the number of linearly independent rows is 3 and the number of linearly independent columns is also 3. This means that the rank of the matrix A is 3 and the matrix A is full-rank.

4 Solving a System of Linear Equations Using the Null Space

The null space (or kernel) of a matrix is playing a critical role in finding solutions to a system of linear equations. Consider a system of linear equations in the matrix form:

$$AX = B$$

where A is the matrix of coefficients, X is the vector of variables, and B is the constant vector.

The null space of an $n \times m$ matrix A, denoted as $N(A)$, is the set of all vectors X that satisfy the equation $AX = \mathbf{0}$:

$$N(A) = \{X \in R^m \mid AX = 0\}.$$

In the context of solving the system of linear equations $AX = B$, the null space of the coefficient matrix A provides insights into the nature of the solutions to the system.

4.1 Fundamental Properties of Null Space

1. The zero vector is always an element of $N(A)$. This is because multiplying any matrix A by the zero vector yields the zero vector: $A0 = 0$.

2. For an $n \times n$ invertible matrix A, the only solution to the equation $AX = 0$ is the trivial solution, $X = 0$.

   proof. $AX = 0 \quad \Rightarrow \quad A^{-1}AX = 0 \quad \Rightarrow \quad I_nX = 0 \quad \Rightarrow \quad X = 0 \quad \Rightarrow \quad N(A) = \{0 \in R^n\}$

3. For the $n \times m$ zero matrix A, the null space equates to the entire vector space:

   $$N(A) = \{X \in R^m | AX = 0\} = R^m.$$

4. If $\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_n$ are vectors in $N(A)$, then:

   $$span(\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_n) \subseteq N(A).$$

   proof. Let $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$ be any vector in the span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. We have:

   $$A\mathbf{v} = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n) = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \ldots + c_nA\mathbf{v}_n = 0.$$

   Therefore, $\mathbf{v} \in N(A)$, and hence, $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) \subseteq N(A)$.

5. If $N(A)$ is nontrivial (i.e., contains more than just the zero vector), the solution to $AX = B$ is not unique.

   proof. Given that $N(A)$ is nontrivial, there exists at least one non-zero vector $\mathbf{V}$ such that $A\mathbf{V} = 0$. If X_p is a particular solution to $AX = B$, then $AX_p = B$. Consider another vector $X' = X_p + \mathbf{V}$. We have:

   $$AX' = A(X_p + \mathbf{V}) = AX_p + A\mathbf{V} = B + 0 = B.$$

   This shows X' is also a solution to $AX = B$. Hence, the solution to $AX = B$ is not unique when $N(A)$ is nontrivial.

6. The null space of A is indeed the same as the null space of U, in an LU decomposition.

   proof. If LU is a factorisation of A, sinc L is invertible, we have:

   $$AX = 0 \quad \Leftrightarrow \quad LUX = 0 \quad \Leftrightarrow \quad UX = 0$$

   Therefore, $N(A)$ and $N(U)$ are equal.

7. The null space of any matrix A is a subspace of $\mathbb{R}^n$. This means that it is closed under vector addition and scalar multiplication. If $\mathbf{v}$ and $\mathbf{w}$ are in $N(A)$, then so is $\mathbf{v} + \mathbf{w}$ and any scalar multiple $c\mathbf{v}$.

8. The dimension of the null space of a matrix, also known as the nullity of the matrix, is a fundamental characteristic of the matrix. The nullity of A is the number of vectors in a basis for $N(A)$. A basis for the null space is a set of linearly independent vectors that span the null space. It can be found by reducing A to its row-echelon form using Gaussian elimination and solving the system $AX = \mathbf{0}$. The free variables resulting from this system correspond to the basis vectors.

9. The Rank-Nullity Theorem is a fundamental theorem in linear algebra that relates the rank and nullity of a matrix. It states that the rank of A (number of pivot columns) plus the nullity of A is equal to the number of columns in A: $\text{rank}(A) + \text{nullity}(A) = n$. This theorem provides a direct link between the column space and the null space of a matrix.

4.2 The Null Space and Solutions

In conclusion, understanding the null space of a matrix and its properties is essential when working with systems of linear equations, influencing whether a system has a unique solution, infinitely many solutions, or no solution at all.

Unique Solution: When the null space of A contains only the zero vector (trivial null space), the system of equations has exactly one solution. This generally occurs when A is an invertible or full-rank matrix.

Infinite Solutions: If the null space of A is nontrivial (contains vectors other than the zero vector), the system has infinitely many solutions. For any particular solution $\mathbf{x}_p$, every vector in the null space can be added to X_p to form another valid solution. In this scenario, the null space vectors act as "degrees of freedom" in the solution to the system of equations.

No Solution: If the constant vector B is not in the column space of A, the system has no solutions. This occurs when the system of equations is inconsistent, often detected during Gaussian elimination when an impossible equation such as $0 = c$ (where c is non-zero) is derived.

5 Utilizing the Null Space of a Matrix to Solve Systems of Linear Equations

Now, we are ready to tackle a system of linear equations, represented as $AX = B$, by utilizing the concept of the null space of a matrix. This method is particularly useful when dealing with systems that have multiple solutions or when the system is underdetermined, i.e., there are more variables than equations.

To employ this method, we first express our system of linear equations in an augmented matrix form, incorporating the coefficients of variables and constants on the right side of the equations into a single matrix. We then apply row operations to transform the matrix into its reduced row-echelon form, that will reveal the pivot variables (corresponding to leading ones in the rows) and free variables (corresponding to all-zero rows).

Pivot Variables: Correspond to the columns in the row-reduced matrix that contain a leading 1 (pivot). These variables can be determined uniquely by the system and are not free to take any value. Their values depend on the free variables of the system.

Free Variables: Correspond to the columns without leading 1's in the row-reduced matrix. These variables can take any value in the field of the system (usually the real numbers). The values of free variables are not restricted by the system of equations.

The pivot variables provide us with a particular solution when we set all the free variables to zero. This particular solution, however, only represents one of potentially many solutions.

This is where the null space comes into play. The vectors in this space are solutions to the homogeneous system $AX = 0$, and these solutions form the basis for the null space.

Ultimately, the solution set to the system $AX = B$ is the sum of the particular solution and the null space. This comprehensive solution set incorporates both the specific solution (found from the pivot variables) and all possible solutions arising from the free variables. Therefore, by using the null space of a matrix, we can fully analyze and solve our system of linear equations.

Example 5.1 *Consider the following system of linear equations:*

$$\begin{cases} x + 2y + 3z = 10 \\ 2x + 4y + 6z = 20 \\ 3x + 6y + 9z = 30 \end{cases}$$

The system can be compactly represented using an augmented matrix. The matrices A, X, and B are defined as follows:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}.$$

We then transform the augmented matrix A to its reduced row-echelon form, U:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 2 & 4 & 6 & 20 \\ 3 & 6 & 9 & 30 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here, the rank of matrix A (represented as $rank(A)$) is equal to the rank of its reduced row-echelon form. Hence, $rank(A) = 1$, which implies that the nullity of A (the dimension of the null space) is 2. In this system, x is the leading variable (corresponding to the first nonzero row), while y and z are free variables (associated with the zero rows). Therefore, y and z can be any real numbers, and the value of x depends on y and z. Setting $y = z = 0$ yields a specific solution to $AX = B$, which we denote as X_p:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x + 2y + 3z = 10 \xrightarrow[y=0]{z=0} x = 10,$$

which gives $X_p = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$.

The null space of A, $N(A)$ or equivalently $N(U)$, is defined as the set of vectors X that satisfy $UX = 0$:

$$N(A) = N(U) = \left\{X \in \mathbb{R}^3 | UX = 0\right\}$$

This leads us to solve the system

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which simplifies to $x + 2y + 3z = 0$, and hence to $x = -2y - 3z$.

The vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ will lie in $N(U)$ if $x = -2y - 3z$, with no restrictions on y and z. Thus,

$$N(A) = \left\{ \begin{bmatrix} -2y - 3z \\ y \\ z \end{bmatrix} \middle| \; y, z \in \mathbb{R} \right\}$$

We can express this in terms of a basis for $N(A)$:

$$N(A) = \left\{ y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \middle| \; y, z \in \mathbb{R} \right\}$$

Hence, $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ form a basis for $N(A)$.

The solution set of the system $AX = B$ is the sum of the particular solution and the null space, given by $X = X_p + N(A)$:

$$X_p + N(A) = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} + \left\{ a \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \middle| a, b \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} + a \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$$

Therefore, the solution set consists of the particular solution X_p and all vectors in the null space of A.

Example 5.2 *Consider the following system of linear equations:*

$$\begin{cases} x - y + z = 2 \\ 2x + 3y - z = 5 \\ 4x + 6y - 2z = 10 \end{cases}$$

This system can be represented as an augmented matrix, with A, X, and B defined as follows:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & -1 \\ 4 & 6 & -2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}.$$

Transforming the augmented matrix A to its reduced row-echelon form, U:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & 3 & -1 & 5 \\ 4 & 6 & -2 & 10 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 5 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The rank of A, is $rank(A) = 2$ and $nullity(A) = 1$. In this system, x and y are pivot variables, while z is a free variable. Thus, z can take any value, while x and y are determined by z.

A particular solution for $AX = B$, denoted as X_p, can be found by setting $z = 0$:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 5 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x - y + z = 2 \\ 5y - 3z = 1 \end{cases} \xRightarrow{z=0} \begin{cases} x = 2.2 \\ y = 0.2 \end{cases},$$

which gives $X_p = \begin{bmatrix} 2.2 \\ 0.2 \\ 0 \end{bmatrix}$. The null space of A, $N(A)$ or equivalently $N(U)$, is defined as the set of vectors X that satisfy $UX = 0$:

$$N(A) = N(U) = \{X \in \mathbb{R}^3 | UX = 0\}$$

This leads us to solve the system

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 5 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which simplifies to $x - y + z = 0$ and $5y - 3z = 0$, and hence to $x = -\frac{2}{5}z$ and $y = \frac{3}{5}z$.

The vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ will lie in $N(U)$ if $x = -\frac{2}{5}z$ and $y = \frac{3}{5}z$, with no restrictions on z. Thus,

$$N(A) = \left\{ \begin{bmatrix} -\frac{2}{5}z \\ \frac{3}{5}z \\ z \end{bmatrix} \middle| z \in \mathbb{R} \right\}$$

We can express this in terms of a basis for $N(A)$:

$$N(A) = \left\{ \frac{1}{5}z \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} \middle| z \in \mathbb{R} \right\}$$

Hence, $\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$ forms a basis for $N(A)$.

The solution set of the system $AX = B$ is the sum of the particular solution and the null space, given by $X = X_p + N(A)$.

$$X_p + N(A) = \begin{bmatrix} 2.2 \\ 0.2 \\ 0 \end{bmatrix} + \left\{ a \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} \middle| a \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 2.2 \\ 0.2 \\ 0 \end{bmatrix} + a \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} \middle| a \in \mathbb{R} \right\}$$

Therefore, the solution set consists of the particular solution X_p and all vectors in the null space of A.