Gaussian Elimination

Wednesday 02.08.2023 (Room M1)

1 Overview

A system of linear equations consists of multiple linear equations with the same variables. Each equation in the system represents a linear relationship between the variables, and the goal is to find values for the variables that satisfy all the equations simultaneously. For instance, let's consider a system of three equations:

$$\begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \\ a_3x + b_3y + c_3z = d_3. \end{cases}$$

The variables are x, y, and z, and for $i = 1, 2, 3$ the coefficients a_i, b_i, and c_i represent the weights or scaling factors for each variable. The constants d_i, $(i = 1, 2, 3)$, on the right-hand side of the equations represent the outcomes or values that the left-hand side should equal.

1.1 Coefficient Matrix and Augmented Matrix

To analyze and solve a system of equations, we often use matrices to represent the system in a concise form. Two important matrices associated with a system of equations are the coefficient matrix and the augmented matrix. The coefficient matrix contains only the coefficients of the variables in the system. On the other hand, the augmented matrix includes both the coefficients of the variables and the constants from the right-hand side of the equations. The matrix of equations for the given system is:

$$\begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \\ a_3x + b_3y + c_3z = d_3. \end{cases} \quad \Leftrightarrow \quad \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

The coefficient matrix is:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

And the augmented matrix for the system is:

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

These matrices represent the system of three equations in matrix form, with the variables x, y, and z represented as a column vector, and the constants on the right-hand side as a separate column vector in the augmented matrix.

Example 1.1 *For the following example:*

$$\begin{cases} 2x + 3y - z = 5, \\ 4x - y + z = 6, \\ -2x + 5y + 2z = 7. \end{cases}$$

The corresponding coefficient matrix is:

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & -1 & 1 \\ -2 & 5 & 2 \end{bmatrix}.$$

For the system, the augmented matrix is:

$$A|B = \left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & -1 & 1 & 6 \\ -2 & 5 & 2 & 7 \end{array}\right].$$

In summary, the coefficient matrix contains only the coefficients of the variables, while the augmented matrix includes both the coefficients and the constants from the right-hand side of the equations. These matrices serve as valuable tools in solving systems of linear equations and analyzing their properties.

2 Matrix Representation and Row-Echelon Form

Consider the following system of equations:

$$\begin{cases} 4x + 3y - z + w + 5u - v = 20 \\ 2y - 2z + 5w + 6u - 3v = -3 \\ z + 3w + 4u - 5v = -10 \\ w + 3u + 4v = 11 \\ u - 3v = -4 \\ 2v = 4 \end{cases} \quad \Rightarrow \begin{cases} v = 2, \\ u = 2, \\ w = -3, \\ z = 1, \\ y = 4, \\ x = 1, \end{cases}$$

In the given system of equations, we can proceed directly with back substitution to determine the values of the variables step by step. But we can transforming every system of equation to this form, that is called row-echelon form.

2.1 Row-Echelon Form and Reduced Row-Echelon Form

Row-Echelon Form: A matrix is said to be in row-echelon form if it satisfies the following conditions:

1. The leading coefficient (also known as the pivot) of any nonzero row is always strictly to the right of the leading coefficient of the row above it.
2. All entries below a leading entry in each column are zeros.

3. Any zero rows, if present, are located at the bottom of the matrix.

Example 2.1 *Consider the following example matrix in row-echelon form:*

$$\begin{bmatrix} \color{blue}{2} & 1 & 3 & 5 \\ 0 & \color{blue}{1} & -1 & 2 \\ 0 & 0 & 0 & \color{blue}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, let's discuss each condition in more detail using the example matrix:

- *In our example, the leading coefficients (pivots) are highlighted in blue:* 2, 1, *and* 3. *As we move from top to bottom, each nonzero row has its leading coefficient to the right of the leading coefficient of the row above it.*
- *All the entries below each leading entry (pivot) in each column are zeros. For instance, below the leading entry 2 in the first column, all entries in that column are zeros. This property holds for each leading entry in row-echelon form.*
- *The last row consists entirely of zeros and is positioned at the bottom.*

Reduced Row-Echelon Form: A matrix is in reduced row echelon form (also called row canonical form) if it satisfies the following conditions:

1. It is in row echelon form.
2. The leading entry in each nonzero row is a 1 (called a leading 1).
3. Each column containing a leading 1 has zeros in all its other entries.

Example 2.2 *Consider the following example matrix in row-echelon form:*

$$\begin{bmatrix} \color{blue}{1} & 0 & 3 & 0 \\ 0 & \color{blue}{1} & -1 & 0 \\ 0 & 0 & 0 & \color{blue}{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, let's discuss each condition in more detail using the example matrix:

- *Each nonzero row has its leading coefficient to the right of the leading coefficient of the row above it, all the entries below each leading entry (pivot) in each column are zeros and zero row is positioned at the bottom. Therefore the matrix is in echelon form.*
- *The leading entry in each nonzero row is a 1.*
- *Each column containing a leading 1 has zeros in all its other entries.*

Elementary Row Operations

When solving a system of equations, several operations, known as elementary row operations, can be performed on the equations without altering the solutions. These operations include:

1. **Interchanging Equations**: Two equations in the system can be interchanged without affecting the solutions. This operation allows for reordering the equations or changing their positions in the system.

2. **Scaling Equations**: An equation in the system can be multiplied by a non-zero constant. This operation allows for scaling the coefficients in an equation, which does not change the solutions of the system.

3. **Adding or Subtracting Equations**: One equation can be added to or subtracted from another equation in the system. This operation enables the combination of equations to eliminate variables or simplify the system. The solutions remain the same.

To obtain the row-echelon form of a matrix, we apply elementary row operations systematically until we achieve the desired form. Once we have the row-echelon form, we can further manipulate the matrix using elementary row operations to obtain the reduced row echelon form.

2.2 Gaussian Elimination and Gauss-Jordan Elimination

Gaussian Elimination: Gaussian elimination is a systematic method for solving a system of linear equations by transforming the augmented matrix of the system into row-echelon form. It involves applying elementary row operations to eliminate variables one by one until the system is simplified into an upper triangular form. The resulting row-echelon form allows us to solve the system of equations more easily by back substitution.

$$\begin{array}{c}\text{Augmented Matrix of}\\ \text{A System of Linear Equations}\end{array} \xrightarrow[\text{Using Elementary Row Operations}]{\text{Gaussian Elimination}} \begin{array}{c}\text{Row-Echelon Form of}\\ \text{the Augmented Matrix}\end{array}$$

Gauss-Jordan Elimination: Gauss-Jordan elimination is an extension of Gaussian elimination. It aims to transform the augmented matrix into reduced row echelon form. Similar to Gaussian elimination, Gauss-Jordan elimination involves applying elementary row operations to the augmented matrix. However, instead of stopping at row-echelon form, Gauss-Jordan elimination continues the elimination process until the matrix reaches reduced row echelon form. This form provides a unique and simplified solution to the system of equations, if one exists, or indicates the inconsistency of the system.

$$\begin{array}{c}\text{Augmented Matrix of}\\ \text{A System of Linear Equations}\end{array} \xrightarrow[\text{Using Elementary Row Operations}]{\text{Gauss-Jordan Elimination}} \begin{array}{c}\text{Reduced Row-Echelon Form of}\\ \text{the Augmented Matrix}\end{array}$$

Both Gaussian elimination and Gauss-Jordan elimination start with augmented matrices, which are formed by combining the coefficient matrix and the constant vector of a system of linear equations. These methods allow us to efficiently manipulate the augmented matrix using elementary row operations to obtain a simplified form that facilitates the solution of the system or reveals its properties.

Example 2.3 *Consider the following system of equations:*

$$\begin{cases} 2x + y - z + w = 4, \\ x - 3y + 2z - w = 9, \\ 3x + 2y - 2z + 4w = 7, \\ 4x + y + 3z - 2w = 15. \end{cases}$$

We will use Gaussian elimination to transform the augmented matrix of this system into row-echelon form.

Step 1: The augmented matrix of the system is:

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 4 \\ 1 & -3 & 2 & -1 & 9 \\ 3 & 2 & -2 & 4 & 7 \\ 4 & 1 & 3 & -2 & 15 \end{array}\right]$$

Step 2: We'll use the first row as a pivot row and perform row operations to eliminate the coefficients below the pivot. We'll subtract $\frac{1}{2}$ times the first row from the second row, $\frac{3}{2}$ times the first row from the third row and 2 times the first row from the fourth row.

$$-\frac{1}{2}\ row_1 + row_2 \quad \left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 4 \\ 1 & -3 & 2 & -1 & 9 \\ 3 & 2 & -2 & 4 & 7 \\ 4 & 1 & 3 & -2 & 15 \end{array}\right] = \left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 4 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{3}{2} & 7 \\ 3 & 2 & -2 & 4 & 7 \\ 4 & 1 & 3 & -2 & 15 \end{array}\right]$$

$$\text{-}\frac{3}{2}\ row_1 + row_3 \quad \left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 4 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{3}{2} & 7 \\ 3 & 2 & -2 & 4 & 7 \\ 4 & 1 & 3 & -2 & 15 \end{array}\right] = \left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 4 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{3}{2} & 7 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{5}{2} & 1 \\ 4 & 1 & 3 & -2 & 15 \end{array}\right]$$

$$-2\ row_1 + row_4 \quad \left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 4 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{3}{2} & 7 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{5}{2} & 1 \\ 4 & 1 & 3 & -2 & 15 \end{array}\right] = \left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 4 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{3}{2} & 7 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{5}{2} & 1 \\ 0 & -1 & 5 & -4 & 7 \end{array}\right]$$

Step 3: We'll use the second row as a pivot row and perform row operations to eliminate the coefficients below the pivot. We'll add $\frac{1}{7}$ times the second row to the third row and $-\frac{2}{7}$ times the second row to the fourth row.

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 4 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{3}{2} & 7 \\ 0 & 0 & -\frac{1}{7} & \frac{16}{7} & 2 \\ 0 & 0 & \frac{30}{7} & -\frac{25}{7} & 5 \end{array}\right]$$

Step 4: We'll use the third row as a pivot row and perform row operations to eliminate the coefficient below the pivot.

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 4 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{3}{2} & 7 \\ 0 & 0 & -\frac{1}{7} & \frac{16}{7} & 2 \\ 0 & 0 & 0 & 65 & 65 \end{array}\right]$$

The resulting matrix is now in row-echelon form, with the pivots highlighted in red. After performing Gaussian elimination and reaching the row-echelon form, we can now utilize back substitution for the following system of equations to find the values of the variables.

$$\begin{cases} 2x + y - z + w = 4, \\ -\frac{7}{2}y + \frac{5}{2}z - \frac{3}{2}w = 7, \\ -\frac{1}{7}z + \frac{16}{7}w = 2, \\ 65w = 65. \end{cases}$$

The solution to the system of equations is $x = 3$, $y = -1$, $z = 2$, and $w = 1$.

Example 2.4 *Consider the following system of equations:*

$$\begin{cases} 2y + z = 4, \\ 2x + y - z = 1, \\ 3x - y + 2z = 5. \end{cases}$$

To transform the augmented matrix of this system into row-echelon form, we'll perform row operations. However, the first step requires swapping rows since the first row has all zeros in the leading position (pivot). Swapping rows allows us to bring a nonzero entry to the leading position.

The augmented matrix for the system is:

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 4 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & 2 & 5 \end{array}\right]$$

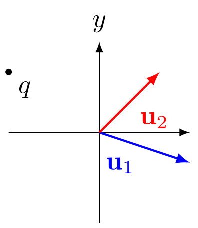
Let's swap the first and second rows:

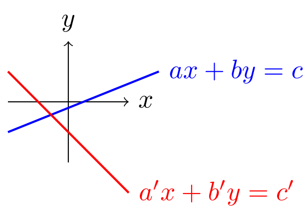
$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 2 & 1 & 4 \\ 3 & -1 & 2 & 5 \end{array}\right]$$

Now, we can continue performing row operations to further reduce the matrix into row-echelon form if needed.

Example 2.5 *1. Unique Solution: A system of equations, and the corresponding row-echelon and reduced ow-echelon forms of its augmented matrix:*

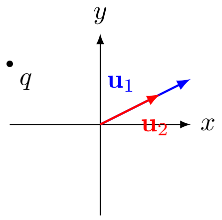
$$\begin{cases} 2x + 3y = 5, \\ 4x - 2y = 6. \end{cases} \Rightarrow \left[\begin{array}{cc|c} 2 & 3 & 5 \\ 4 & -2 & 6 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & -8 & -4 \end{array}\right], \quad \left[\begin{array}{cc|c} 1 & 0 & \frac{7}{4} \\ 0 & 1 & \frac{1}{2} \end{array}\right]$$

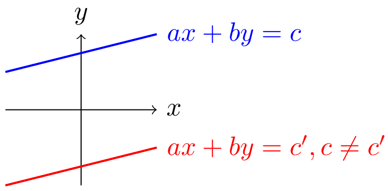




2. *No Solution: A system of equations, and the corresponding row-echelon and reduced ow-echelon forms of its augmented matrix:*

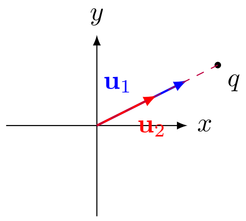
$$\begin{cases} 2x + 3y = 5, \\ 4x + 6y = 12. \end{cases} \quad \Rightarrow \quad \left[\begin{array}{cc|c} 2 & 3 & 5 \\ 4 & 6 & 12 \end{array}\right] \quad \Rightarrow \quad \left[\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & 0 & 2 \end{array}\right], \quad \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & 0 & 2 \end{array}\right]$$

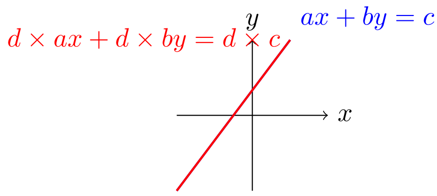




3. *Infinite Solutions: A system of equations, and the corresponding row-echelon and reduced ow-echelon forms of its augmented matrix:*

$$\begin{cases} 2x + 3y = 5, \\ 4x + 6y = 10. \end{cases} \quad \Rightarrow \quad \left[\begin{array}{cc|c} 2 & 3 & 5 \\ 4 & 6 & 10 \end{array}\right] \quad \Rightarrow \quad \left[\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & 0 & 0 \end{array}\right]$$





Example 2.6 *Consider the following matrix A. We will find A^{-1} using Gauss-Jordan elimination:*

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 1 \\ 4 & -2 & 1 \end{bmatrix}$$

To find the inverse of A, since $AA^{-1} = I$ we can augment A with the identity matrix I:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 & 1 & 0 \\ 4 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

Next, we apply elementary row operations to transform the augmented matrix into reduced row echelon form.

1. *Multiply the first row by $1/2$ to create a leading 1:*

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ -1 & 3 & 1 & 0 & 1 & 0 \\ 4 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

2. *Add the first row to the second row:*

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{7}{2} & 1 & \frac{1}{2} & 1 & 0 \\ 4 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

3. Subtract 4 *times the first row from the third row:*

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{7}{2} & 1 & \frac{1}{2} & 1 & 0 \\ 0 & -4 & 1 & -2 & 0 & 1 \end{array}\right]$$

4. Multiply the second row by $2/7$ *to create a leading 1:*

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{2}{7} & \frac{1}{7} & \frac{2}{7} & 0 \\ 0 & -4 & 1 & -2 & 0 & 1 \end{array}\right]$$

5. Add 4 *times the second row to the third row:*

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{2}{7} & \frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & \frac{15}{7} & -\frac{10}{7} & \frac{8}{7} & 1 \end{array}\right]$$

6. Multiply the third row by $7/15$ *to create a leading 1:*

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{2}{7} & \frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{8}{15} & \frac{7}{15} \end{array}\right]$$

7. Subtract $\frac{2}{7}$ *times the third row from the second row:*

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{2}{15} & -\frac{2}{15} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{8}{15} & \frac{7}{15} \end{array}\right]$$

8. Subtract $\frac{1}{2}$ *times the second row from the first row:*

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{15} & \frac{1}{15} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{2}{15} & -\frac{2}{15} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{8}{15} & \frac{7}{15} \end{array}\right]$$

Thus, the inverse of matrix A *is given by:*

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{15} & \frac{1}{15} \\ \frac{1}{3} & \frac{2}{15} & -\frac{2}{15} \\ -\frac{2}{3} & \frac{8}{15} & \frac{7}{15} \end{bmatrix} = \frac{1}{15} \begin{bmatrix} 5 & -1 & 1 \\ 5 & 2 & -2 \\ -10 & 8 & 7 \end{bmatrix}$$