Matrices

Monday 31.07.2023 (Room M1)

1 Overview

A matrix is a two-dimensional array of numbers, symbols, or expressions arranged in rows and columns. Matrices play a fundamental role in various areas of mathematics, such as linear algebra, statistics, computer science, and physics. They provide a convenient way to organize and manipulate data. Matrices support various operations, including addition, subtraction, scalar multiplication, matrix multiplication, and transpose. Also, the matrix inverses have numerous applications in solving systems of linear equations, computing solutions to linear transformations, and performing various matrix operations.

Matrices are typically denoted by uppercase letters, such as A, B, or C. The dimensions of a matrix are specified by the number of rows and columns it contains. For example, a matrix with m rows and n columns is referred to as an $m \times n$ matrix. The individual elements of a matrix are identified by their position using subscripts. For instance, a_{ij} represents the element in the i-th row and j-th column of matrix A.

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

1.1 Transpose of a Matrix

The transpose of a matrix A, denoted as A^T, is obtained by interchanging its rows with columns. In other words, the i-th row of matrix A becomes the i-th column of A^T, and the j-th column of matrix A becomes the j-th row of A^T.

Example 1.1 *Consider the matrix*

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

The transpose of M is

$$M^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

1.2 Types of Matrices

1. **Square Matrix**: A square matrix has an equal number of rows and columns (i.e., $m = n$).

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

2. **Row Matrix**: A row matrix has a single row and multiple columns (i.e., $m = 1$). The transpose of a row matrix simply converts it into a column matrix:

$$B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \qquad B^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

3. **Column Matrix**: A column matrix has a single column and multiple rows (i.e., $n = 1$).The transpose of a column matrix converts it into a row matrix:

$$C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad C^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

4. **Zero Matrix**: A zero matrix, denoted by the symbol O, is a matrix in which all the elements are zero.

$$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5. **Identity Matrix**: An identity matrix, denoted by the symbol I, is a square matrix with ones on the main diagonal (from the top left to the bottom right) and zeros elsewhere. The transpose of an identity matrix remains the same as the original matrix:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad I^T = I$$

6. **Diagonal Matrix**: A diagonal matrix is a square matrix in which all the non-diagonal elements are zero. The transpose of a diagonal matrix remains the same as the original matrix:

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \qquad D^T = D$$

7. **Symmetric Matrix**: A symmetric matrix is a square matrix that remains unchanged when its rows are converted into columns and vice versa. The transpose of a symmetric matrix remains the same as the original matrix:

$$S = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}, \qquad S^T = S$$

8. **Skew-Symmetric Matrix**: A skew-symmetric matrix is a square matrix that satisfies the condition $R^T = -R$, where R^T denotes the transpose of matrix R. The transpose of a skew-symmetric matrix changes the sign of each element:

$$R = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -5 \\ -3 & 5 & 0 \end{bmatrix}, \qquad R = -R^T$$

9. **Upper Triangular Matrix**: An upper triangular matrix is a square matrix in which all the elements below the main diagonal (from top left to bottom right) are zero. The transpose of a upper triangular matrix converts it into a lower triangular matrix:

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \qquad U^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}$$

10. **Lower Triangular Matrix**: A lower triangular matrix is a square matrix in which all the elements above the main diagonal (from top left to bottom right) are zero. The transpose of a lower triangular matrix converts it into a upper triangular matrix:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix} \qquad L^T = \begin{bmatrix} 1 & 4 & 7 \\ 0 & 5 & 8 \\ 0 & 0 & 9 \end{bmatrix}$$

1.3 Matrix Addition

Matrices can be added together if they have the same dimensions. The addition is performed by adding corresponding elements of the matrices.

$$A + B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}$$

1.4 Expressing Matrix Multiplication using Dot Product of Vectors

Matrix multiplication can be defined using the dot product of vectors. Consider matrices A and B, where A has dimensions $m \times n$ and B has dimensions $n \times p$. We can express the resulting matrix C as:

$$AB = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix} = \begin{bmatrix} \xrightarrow{\mathbf{a}_1} \\ \xrightarrow{\mathbf{a}_2} \\ \vdots \\ \xrightarrow{\mathbf{a}_m} \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \big\downarrow & \mathbf{b}_2 \big\downarrow & \dots & \mathbf{b}_p \big\downarrow \end{bmatrix} = \begin{bmatrix} a_1^T \cdot b_1 & a_1^T \cdot b_2 & \dots & a_1^T \cdot b_p \\ a_2^T \cdot b_1 & a_2^T \cdot b_2 & \dots & a_2^T \cdot b_p \\ \vdots & \vdots & \ddots & \vdots \\ a_m^T \cdot b_1 & a_m^T \cdot b_2 & \dots & a_m^T \cdot b_p \end{bmatrix}$$

where $\mathbf{a}_i$ represents the i-th row vector of matrix A and $\mathbf{b}_j$ represents the j-th column vector of matrix B.

1.5 Scalar Multiplication

A matrix can be multiplied by a scalar (a single number). This operation is performed by multiplying each element of the matrix by the scalar.

$$k \cdot A = k \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

where k is a real number.

1.6 Matrix Inverse

Given a square matrix A, the inverse of A, denoted as A^{-1}, is a matrix such that:

$$AA^{-1} = A^{-1}A = I,$$

where I is the identity matrix. In other words, multiplying a matrix by its inverse results in the identity matrix. A matrix that has inverse is called non-singular or invertible.

Example 1.2 *Given a 2×2 matrix A:*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the inverse of A, denoted as A^{-1}, can be calculated as follows:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

It is important to note that the inverse of a 2×2 matrix exists only if $ad - bc$ (the determinant) is non-zero.

Example 1.3 *To rotate a point $\begin{bmatrix} x \\ y \end{bmatrix}$ counterclockwise by an angle θ, we can multiply the coordinates of the point by the rotation matrix:*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where the vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ represent the rotated point. The reverse of the rotation matrix $R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ is:

$$R^{-1} = \frac{1}{\cos(\theta) \cdot \cos(\theta) - (-\sin(\theta) \cdot \sin(\theta))} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

But,

$$R^{-1} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

We can observe that the reverse of the rotation matrix is the same as rotating by $-\theta$ will bring the point back to its original position. In other words, rotating by $-\theta$ is equivalent to applying the reverse of the rotation matrix.

1.7 Properties of Matrices

Matrices have several properties that are important to understand:

1. **Commutativity of Addition**: $A + B = B + A$
2. **Associativity of Addition**: $(A + B) + C = A + (B + C)$
3. **Distributivity of Scalar Multiplication**: $k \cdot (A + B) = k \cdot A + k \cdot B$

4. **Distributivity of Scalar Multiplication**: $k \cdot (AB) = (k \cdot A)B = A(k \cdot B)$
5. **Associativity of Matrix Multiplication**: $(AB)C = A(BC)$
6. **Non-commutativity of Matrix Multiplication**: $AB \neq BA$ (in general)
7. **Identity Matrix**: $AI = IA = A$
8. **Zero Matrix**: $A + O = O + A = A$
9. **Transpose of Transpose**: $(A^T)^T = A$
10. **Transpose of a Sum**: $(A + B)^T = A^T + B^T$
11. **Transpose of a Scalar Multiple**: $(k \cdot A)^T = k \cdot A^T$
12. **Transpose of a Product**: $(AB)^T = B^T A^T$ (Note: The order is reversed)
13. **Symmetry of** $(AA^T)^T$: $(AA^T)^T = AA^T$ ($(AA^T)^T$is a symmetric matrix)
14. **Inverse of Inverse**: $(A^{-1})^{-1} = A$
15. **Inverse of a Product**: $(AB)^{-1} = B^{-1}A^{-1}$ (Note: The order is reversed)
16. **Inverse of a Transpose**: $(A^T)^{-1} = (A^{-1})^T$

Example 1.4 *We want to verify the matrix property $(AB)^T = B^T A^T$, using the following 2×2 matrices:*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.$$

The product of A and B, (AB) is:

$$AB = \begin{bmatrix} 1 \times 5 + 2 \times 7 & 1 \times 6 + 2 \times 8 \\ 3 \times 5 + 4 \times 7 & 3 \times 6 + 4 \times 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

The transpose of AB, denoted $(AB)^T$, is obtained by swapping the rows and columns of AB:

$$(AB)^T = \begin{bmatrix} 19 & 43 \\ 22 & 50 \end{bmatrix}$$

The transposes of B and A, denoted B^T and A^T, are:

$$B^T = \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

The product of B^T and A^T is:

$$B^T A^T = \begin{bmatrix} 5 \times 1 + 7 \times 2 & 5 \times 3 + 7 \times 4 \\ 6 \times 1 + 8 \times 2 & 6 \times 3 + 8 \times 4 \end{bmatrix} = \begin{bmatrix} 19 & 43 \\ 22 & 50 \end{bmatrix}$$

Thus, we find that $(AB)^T$ is equal to $B^T A^T$. This property states that the transpose of the product of two matrices A and B is equal to the product of the transposes of the matrices B and A, but importantly, in reverse order.

Example 1.5 *To prove that the inverse of a transpose matrix is equal to the transpose of the inverse matrix $(A^T)^{-1} = (A^{-1})^T$, we will need to demonstrate that the left-hand side and the right-hand side are both inverse matrices of A^T.*

1. *For the left-hand side, $(A^T)^{-1}A^T$:*

$$(A^T)^{-1}A^T = I$$

where I is the identity matrix.

2. *For the right-hand side, $(A^{-1})^T A^T$:*

$$(A^{-1})^T A^T = (AA^{-1})^T$$

We know that AA^{-1} is the identity matrix, so:

$$(A^{-1})^T A^T = (I)^T = I$$

Therefore, both sides multiply by A^T to give the identity matrix, which shows that $(A^T)^{-1}$ and $(A^{-1})^T$ are indeed inverse matrices of A^T, proving that $(A^T)^{-1} = (A^{-1})^T$. This property is known as the "inverse of a transpose" property and holds for all invertible matrices.

2 Image Transformations Using 2×2 Matrices

Using 2×2 matrices, we can apply a variety of transformations to an image, including translation, rotation, scaling, shearing, and reflection. These transformations allow us to modify the position, size, orientation, and symmetry of an image. Understanding the mathematical representations of these transformations using matrices is crucial in image processing and computer vision applications.

Figure 1: Original Image

2.1 Translation

Translation is the process of shifting an image horizontally and vertically. To perform a translation, we add the translation vector (t_x, t_y) to the coordinates of each pixel in the image:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

Figure 2: Shifted Image

where $\begin{bmatrix} x \\ y \end{bmatrix}$ are the original coordinates of the pixel and $\begin{bmatrix} x' \\ y' \end{bmatrix}$ are the translated coordinates.

2.2 Rotation

Rotation is the process of rotating an image counterclockwise by an angle θ. We can represent rotation using a 2×2 matrix. To perform a rotation, we multiply the coordinates of each pixel in the image by the rotation matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Figure 3: Image with Rotation

where $\begin{bmatrix} x \\ y \end{bmatrix}$ are the original coordinates of the pixel and $\begin{bmatrix} x' \\ y' \end{bmatrix}$ are the rotated coordinates.

2.3 Scaling

Scaling is the process of resizing an image by a scale factor s_x along the x-axis and s_y along the y-axis. We can represent scaling using a 2×2 matrix. To perform scaling, we multiply the scaling matrix by the coordinates of each pixel in the image:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where (x, y) are the original coordinates of the pixel and (x', y') are the scaled coordinates.

Figure 4: Image with Scaling

2.4 Shearing

Shearing is the process of distorting an image along the x-axis and y-axis, in order to adjust or create the impression of viewing it from different angles. It's a common technique used to modify images for computer vision systems. This way, the computer can better understand how humans see things from various perspectives. We can represent shearing using a 2×2 matrix. To perform shearing, we multiply the shearing matrix by the coordinates of each pixel in the image:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 + h_x h_y & h_x \\ h_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where (x, y) are the original coordinates of the pixel and (x', y') are the sheared coordinates.

Figure 5: Image with Shearing

2.5 Reflection

Reflection is the process of mirroring an image across a line. We can represent reflection using a 2×2 matrix. For example, for reflection along the x-axis, we multiply the reflection matrix by the coordinates of each pixel in the image:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Figure 6: Image with Reflection

where (x, y) are the original coordinates of the pixel and (x', y') are the reflected coordinates.

2.6 Affine Transformation

An affine transformation is a combination of geometric transformations such as scaling, rotation, shearing, and reflection, followed by a translation. The transformation is expressed as:

$$T(v) = Av + b. \tag{1}$$

Example 2.1 *Consider a rotation matrix A for 90 degrees counterclockwise:*

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and a translation vector b for moving 2 units to the right and 3 units up:

$$b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Given a vector v:

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

The affine transformation $T(v) = Av + b$ is computed as:

$$T(v) = Av + b = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -v_2 + 2 \\ v_1 + 3 \end{bmatrix}$$

So, the transformed vector $T(v)$ after the affine transformation is $\begin{bmatrix} -v_2 + 2 \\ v_1 + 3 \end{bmatrix}$, which is the result of a 90 degree counterclockwise rotation of the original vector and then a translation.