# Determinant

Monday 14.08.2023 (Room M1)

## 1 Overview

Determinants are mathematical objects associated with square matrices. They have a rich history and play a pivotal role in linear algebra, computational mathematics, and physics.

## 2 Computing Determinants

### 2.1 $2 \times 2$ Matrix

For the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant is:

$$\det(A) = ad - bc$$

### 2.2 $3 \times 3$ Matrix

Expanding by minors:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The determinant is:

$$\det(A) = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

### 2.3 Laplace's Expansion

The determinant of a matrix can be computed using Laplace's expansion. This method involves expanding the determinant across any row or column using the minors and cofactors of the matrix entries. The determinant is calculated as the sum of the products of the entries of the chosen row (or column) and their corresponding cofactors.

**Definition:** Given a matrix $A = (a_{ij})$, the *minor* of an element $a_{ij}$ (denoted $M_{ij}$) is the determinant of the matrix obtained by removing the $i$-th row and $j$-th column from $A$.

The *cofactor* of an element $a_{ij}$ (denoted $\mathrm{cof}(a_{ij})$ ) is given by:

$$\mathrm{cof}(a_{ij}) = (-1)^{i+j} \times M_{ij}$$

**Example 2.1** *Compute the determinant of the matrix:*

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

*Expanding using the top row:*

$$\det(B) = b_{11} \times cof(b_{11}) + b_{12} \times cof(b_{12}) + b_{13} \times cof(b_{13})$$

*Where:*

$$cof(b_{11}) = (-1)^{1+1} \times \det \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix} = 24$$
$$cof(b_{12}) = (-1)^{1+2} \times \det \begin{bmatrix} 0 & 5 \\ 1 & 6 \end{bmatrix} = 5$$
$$cof(b_{13}) = (-1)^{1+3} \times \det \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} = -4$$

*Substituting these values in, we get:*

$$\det(B) = 1(24) + 2(5) - 3(4) = 24 + 10 - 12 = 22$$

*Thus,* $\det(B) = 22$.

*Expanding using the second row:*

$$\begin{aligned} \det(B) &= b_{21} \times cof(b_{21}) + b_{22} \times cof(b_{22}) + b_{23} \times cof(b_{23}) \\ &= 4 \times cof(b_{22}) + 5 \times cof(b_{23}) \end{aligned}$$

*Where:*

$$cof(b_{22}) = (-1)^{2+2} \times \det \begin{bmatrix} 1 & 3 \\ 1 & 6 \end{bmatrix} = 3$$
$$cof(b_{23}) = (-1)^{2+3} \times \det \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = 2$$

*Substituting these values in, we get:*

$$\det(B) = 4(3) + 5(2) = 12 + 10 = 22$$

*Thus,* $\det(B) = 22$.

**Corollary 2.2** *If a matrix $A$ has a row or column consisting entirely of zeros, then* $\det(A) = 0$.

**Proof 2.3** *Using the Laplace expansion (or cofactor expansion), the determinant of a matrix can be expanded along any of its rows or columns. If we choose to expand along a zero row (or column), every term in the expansion is multiplied by a zero from the row (or column). Therefore, every term in the expansion is zero, which implies that the determinant itself is zero.*

*More formally, let's expand the determinant along a zero row. For any element $a_{ij} = 0$ in this row, its contribution to the determinant is:*

$$a_{ij} \times cofactor = 0 \times cofactor = 0$$

*Summing over all such contributions gives:*

$$\det(A) = \sum_{j=1}^{n} 0 = 0$$

*The argument is analogous when expanding along a zero column.*

**Example 2.4** *Consider the matrix:*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

*Here, the second row consists entirely of zeros. Based on our corollary:*

$$\det(A) = 0$$

## 2.4 Upper Triangular Matrix, Lower Triangular Matrix and Diagonal Matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ 0 & a_{22} & a_{23} & \cdots \\ 0 & 0 & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \begin{bmatrix} a_{11} & 0 & 0 & \cdots \\ a_{21} & a_{22} & 0 & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \begin{bmatrix} a_{11} & 0 & 0 & \cdots \\ 0 & a_{22} & 0 & \cdots \\ 0 & 0 & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

For theses matrices, the determinant is the product of its diagonal entries:

$$\det(A) = a_{11} \times a_{22} \times a_{33} \times \ldots$$

## 2.5 LU Decomposition

Another benefit of LU decomposition is finding the determinant. Specifically, when calculating the determinant of $A$, one can leverage the property that the determinant of a triangular matrix is the product of its diagonal entries.

Given that:

$$\begin{aligned} \det(A) &= \det(L \times U) \\ &= \det(L) \times \det(U) \\ &= \det(U) \end{aligned}$$

Since $L$ has diagonal entries all equal to 1, $\det(L) = 1$. Therefore, the determinant of $A$ is equal to the determinant of $U$, which is the product of its diagonal entries.

**Example 2.5** *Consider the matrix:*

$$A = \begin{bmatrix} 4 & 3 \\ 6 & 3 \end{bmatrix}$$

*Applying LU decomposition, we can factorize matrix $A$ into $L$ and $U$ matrices. Here's a potential decomposition:*

$$L = \begin{bmatrix} 1 & 0 \\ 1.5 & 1 \end{bmatrix}$$

*and*

$$U = \begin{bmatrix} 4 & 3 \\ 0 & -1.5 \end{bmatrix}$$

*Notice that multiplying $L$ and $U$ reproduces the matrix $A$:*

$$L \times U = \begin{bmatrix} 1 & 0 \\ 1.5 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 3 \\ 0 & -1.5 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 6 & 3 \end{bmatrix}$$

*To find the determinant of $A$, we can multiply the determinants of $L$ and $U$.*

*Given that:*

$$\det(L) = 1 \times 1 = 1$$

*and*

$$\det(U) = 4 \times (-1.5) = -6$$

*Therefore:*

$$\det(A) = \det(L) \times \det(U) = 1 \times (-6) = -6$$

*Thus, the determinant of matrix $A$ is $-6$.*

## 2.6 Determinant, Linear Independence and Invertibility

**Proposition 2.6** *The determinant of a matrix is zero if and only if its rows (or columns) are linearly dependent.*

**Proof 2.7** *Consider a square matrix $A$ which has an $LU$ factorization such that $A = LU$, where $L$ is a lower triangular matrix and $U$ is an upper triangular matrix.*

***Part 1:*** *If the rows (or columns) of $A$ are linearly dependent, then at least one of the diagonal entries of $U$ (the pivots during the elimination process) will be zero. Thus, $\det(U) = 0$. Since the determinant of a product of matrices is the product of their determinants, $\det(A) = \det(L) \times \det(U) = 0$.*

***Part 2:*** *Conversely, if $\det(A) = 0$, then $\det(L) \times \det(U) = 0$. Since $L$ is a lower triangular matrix with ones on its diagonal, $\det(L) = 1$. Therefore, it must be that $\det(U) = 0$, which means at least one of the pivots (diagonal entries of $U$) is zero, indicating that the rows (or columns) of $A$ are linearly dependent.*

*Thus, we've shown that the determinant of a matrix is zero if and only if its rows (or columns) are linearly dependent.*

**Proposition 2.8** *A square matrix $A$ is invertible (non-singular) if and only if its determinant is non-zero.*

**Proof 2.9** ***Part 1:*** *If $A$ is invertible, then its determinant is non-zero.*

*Suppose, for the sake of contradiction, that $A$ is invertible but $\det(A) = 0$. If $A$ is invertible, then there exists a matrix $B$ such that $AB = BA = I$, where $I$ is the identity matrix. Using properties of determinants, we have:*

$$\det(AB) = \det(A)\det(B)$$

*Given that $AB = I$, $\det(AB) = \det(I) = 1$. However, if $\det(A) = 0$, then $0 \times \det(B) = 1$, which is a contradiction. Hence, $\det(A)$ cannot be zero.*

***Part 2:*** *If the determinant of $A$ is non-zero, then $A$ is invertible.*

*Given that* $\det(A) \neq 0$ *and considering the relationship* $\det(A) = \det(U)$ *in the LU factorization of* $A$*, it follows that* $\det(U) \neq 0$*. This ensures that all the pivots in the factorization are non-zero. Through a sequence of elementary row operations, including potential row scaling, one can transform matrix* $A$ *into the identity matrix* $I$*. In parallel, these same operations can convert* $I$ *into the inverse of* $A$*, denoted* $A^{-1}$*. Consequently, matrix* $A$ *is invertible with inverse* $A^{-1}$*.*

*From the above two parts, we conclude that a matrix is invertible if and only if its determinant is non-zero.*

**Corollary 2.10** *Given a square matrix* $A$*, the following statements are equivalent:*

1. *The determinant of* $A$ *is non zero.*
2. *The rows (or columns) of* $A$ *are linearly independent.*
3. *Matrix* $A$ *is invertible (nonsingular).*

# 3 Properties of Determinants

## 3.1 Basic Properties

**Theorem 3.1** *Let* $A$ *and* $B$ *be matrices of the same size. Then:*

1. $\det(A^T) = \det(A)$ *(Transpose Property)*
2. $\det(AB) = \det(A) \cdot \det(B)$ *(Multiplicative Property)*
3. $\det(A^{-1}) = \frac{1}{\det(A)}$ *if* $A$ *is invertible.*

## 3.2 Effects of Elementary Row Operations

**Theorem 3.2 (Row Operations)** *Given a matrix* $A$ *and its determinant* $\det(A)$*:*

1. *If a row of* $A$ *is multiplied by* $k$*, then the determinant is multiplied by* $k$*.*
2. *Switching two rows of* $A$ *will change the sign of its determinant.*
3. *Adding a multiple of one row to another does not change the determinant.*

**Example 3.3** *Consider the* $2 \times 2$ *matrix* $A$*:*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

***Multiplying a row by a scalar*** *: Now, let's multiply the first row by a scalar* $k$*:*

$$B = \begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$$

*Computing the determinants of both matrices:*

*The determinant of matrix $A$ is:*

$$\det(A) = ad - bc$$

*The determinant of matrix $B$ is:*

$$\det(B) = (ka)d - (kb)c = k(ad - bc) = k \cdot \det(A)$$

*Thus, we see that if a row of matrix $A$ is multiplied by $k$, then the determinant is multiplied by $k$ as well.*

***Switching Two Rows*** *: Switching the two rows results in matrix $C$:*

$$C = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

*Computing the determinants:*

$$\det(A) = ad - bc$$

$$\det(C) = bc - ad = -(ad - bc) = -\det(A)$$

*This confirms that switching two rows of a matrix changes the sign of its determinant.*

***Adding a Multiple of One Row to Another*** *: Consider the same matrix $A$:*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

*Adding $k$ times the first row to the second row results in matrix $D$:*

$$D = \begin{bmatrix} a & b \\ c + ka & d + kb \end{bmatrix}$$

*Computing the determinants:*

$$\det(A) = ad - bc$$

$$\det(D) = ad - bc = \det(A)$$

*This demonstrates that adding a multiple of one row to another does not change the determinant.*

**Example 3.4** *Let $A$ be an $n \times n$ matrix and $k$ be a scalar. The determinant of the matrix obtained by multiplying every entry of $A$ by $k$, denoted as $kA$, is given by:*

$$\det(kA) = k^n \det(A)$$

*This property arises because when you multiply a matrix $A$ by a scalar $k$, you're essentially multiplying each of its rows by $k$. For an $n \times n$ matrix, this results in multiplying the determinant by $k$, $n$ times, leading to the factor of $k^n$.*

*Consider the $2 \times 2$ matrix $A$:*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

*Now, let's multiply $A$ by a scalar $k$:*

$$kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

*The determinant of matrix $kA$ is:*

$$\det(kA) = (ka)(kd) - (kb)(kc) = k^2(ad - bc) = k^2 \cdot \det(A)$$

**Corollary 3.5** *If a matrix has two identical rows (or columns), its determinant is zero.*

**Proof 3.6** *Consider a matrix $A$ and another matrix $A'$ which is obtained by swapping two identical rows of $A$. Since the rows are identical, $A = A'$. However, swapping two rows of a matrix changes the sign of its determinant. Thus, we can write:*

$$\det(A) = -\det(A')$$

*Given that $A = A'$, we have:*

$$\det(A) = -\det(A)$$

*This implies that $2\det(A) = 0$, which leads to:*

$$\det(A) = 0$$

**Example 3.7** *Consider the matrix:*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 4 & 5 & 6 \end{bmatrix}$$

*Here, the second and third rows are identical. Therefore, based on our property:*

$$\det(A) = 0$$

## 3.3 Linearity Property of Determinants

**Property for Columns** : For any matrix $A = [A_1, A_2, \ldots, A_n]$ and $A' = [A_1', A_2, \ldots, A_n]$ where $A_1$ and $A_1'$ are columns of $A$ and $A'$ respectively, the following property holds:

$$\det([A_1 + A_1', A_2, \ldots, A_n]) = \det([A_1, A_2, \ldots, A_n]) + \det([A_1', A_2, \ldots, A_n])$$

**Proof for Columns** : Expanding the determinant of the matrix $B = [A_1 + A_1', A_2, \ldots, A_n]$ along the first column gives:

$$\det(B) = \sum_{i=1}^{n} (-1)^{1+i}(a_{i1} + a_{i1}')M_{i1}$$

This can be split into two sums:

$$\det(B) = \sum_{i=1}^{n} (-1)^{1+i} a_{i1} M_{i1} + \sum_{i=1}^{n} (-1)^{1+i} a_{i1}' M_{i1}$$

Substituting matrices $A$ and $A'$ for the first column of $B$, we obtain:

$$\det(B) = \det(A) + \det(A')$$

This proves the desired property.

**Property for Rows** : Similarly, considering rows instead of columns, for matrices $A$ and $A'$ where the rows are $A_1, A_2, \ldots, A_n$ and $A'_1, A_2, \ldots, A_n$ respectively, we have:

$$\det\left(\begin{bmatrix} A_1 + A'_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}\right) = \det\left(\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}\right) + \det\left(\begin{bmatrix} A'_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}\right)$$

**Proof for Rows** : Given the symmetry property of the determinant with respect to the matrix's transpose, we have:

$$\det(A^T) = \det(A)$$

Expanding the determinant by rows is equivalent to expanding the determinant of the transposed matrix by columns. Therefore, the linearity property with respect to rows is a direct consequence of the column property. By transposing both sides of the column property equation, we obtain the row property.

**Example 3.8** *Let*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

*and*

$$A' = \begin{bmatrix} 5 & 2 \\ 6 & 4 \end{bmatrix}$$

*Then,*

$$B = [A_1 + A'_1, A_2] = \begin{bmatrix} 6 & 2 \\ 9 & 4 \end{bmatrix}$$

*Computing, we get:*

$$\det(A) = (1 \times 4) - (2 \times 3) = -2$$
$$\det(A') = (5 \times 4) - (2 \times 6) = 8$$
$$\det(B) = (6 \times 4) - (2 \times 9) = 6$$

*Verifying the property:*

$$\det(A) + \det(A') = -2 + 8 = 6 = \det(B)$$

# 4 Geometric Interpretation of Determinants

Determinants serve as powerful tools in geometry, offering insights into linear transformations' properties:

- **Orientation:** The determinant's sign sheds light on the orientation. A positive determinant suggests a preservation of orientation, whereas a negative one indicates its reversal. For instance, in 2D, a determinant's sign reveals whether the transformation involves a reflection.
- **Scaling:** The absolute value of a determinant signifies the scaling factor during a transformation in forming how much areas (2D) or volumes (3D) are expanded or contracted.

## 4.1 Orientation: A Closer Look

A crucial application of determinants in geometry is discerning the orientation resulting from a linear transformation.

**Example 4.1 (Reflection in 2D)** *Consider the matrix*

$$T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

*The determinant of $T$ is -1. The negative sign indicates a change in orientation: the transformation mirrors vectors across the $x$-axis.*

**Example 4.2 (Rotation in 2D)** *A rotation matrix in 2D that rotates vectors counter-clockwise by an angle $\theta$ is given by*

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

*The determinant of $R(\theta)$ is always 1, implying that rotations preserve orientation.*

## 4.2 Area of a Parallelogram

The determinant's absolute value of a $2 \times 2$ matrix formed by two vectors in $\mathbb{R}^2$ gives the parallelogram's area spanned by these vectors.

**Example 4.3** *For vectors:*

$$\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

*and*

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

*Computing the area:*

$$A = \left| \det \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \right| = 5$$

*The area is 5 square units.*

## 4.3 Volume of a Parallelepiped

The absolute value of a $3 \times 3$ matrix's determinant, whose columns are the three vectors defining a parallelepiped in $\mathbb{R}^3$, indicates the parallelepiped's volume.

**Example 4.4** *Consider the three vectors forming the edges of a parallelepiped:*

$$\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \quad \textit{and} \quad \mathbf{c} = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

*The volume $V$ of the parallelepiped spanned by these vectors can be found using the determinant of the matrix formed by these vectors as its columns:*

$$V = a.(b \times c) = \left| \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix} \right|$$

*Expanding the determinant along the top row:*

$$\begin{aligned} V &= \left| 1 \times \det \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix} - 2 \times \det \begin{bmatrix} 0 & 5 \\ 1 & 6 \end{bmatrix} + 3 \times \det \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} \right| \\ &= |1 \times (4 \times 6 - 5 \times 0) - 2 \times (0 - 5) + 3 \times (0 - 4)| \\ &= |24 + 10 - 12| = 22. \end{aligned}$$

*Thus, the volume of the parallelepiped defined by the vectors* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *is* 22 *cubic units.*