Span and Linear Independence

Monday 07.08.2023 (Room M1)

1 The Span of Vectors in $\mathbb{R}^n$

In $\mathbb{R}^n$, the span of a set of vectors refers to the set of all possible vectors that can be obtained by taking linear combinations of those vectors.

More formally, given a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in $\mathbb{R}^n$, the span of these vectors, denoted as $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$, is defined as:

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = \left\{ c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k \,\middle|\, c_1, c_2, \ldots, c_k \in \mathbb{R} \right\}$$

The span of the vectors represents all possible vectors that can be formed by scaling and adding the given set of vectors. It includes both the vectors in the original set as well as any linear combinations of those vectors.

Example 1.1 *To find the span of the column vectors* $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ *in* $\mathbb{R}^3$, *we need to determine all possible linear combinations of these vectors. The span represents the set of all vectors that can be obtained by scaling and adding these two column vectors.*

Let's express a generic vector in $\mathbb{R}^3$, (x, y, z), *as a linear combination of the given column vectors:*

$$span(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) = \left\{ a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$$

As you can see, the z-component of the resulting vector is always zero. This means that any vector in the span of the column vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ *will have its z-coordinate fixed at zero.*

In other words, the span of these column vectors in $\mathbb{R}^3$ *is the xy-plane, a two-dimensional subspace within* $\mathbb{R}^3$ *where the z-coordinate is always zero.*

To summarize, the span of the column vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ *in* $\mathbb{R}^3$ *is the xy-plane, which consists of all column vectors of the form* $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$, *where* a *and* b *can be any real numbers.*

Definition 1.2 *Let* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ *be vectors in* $\mathbb{R}^n$.

1. The vectors are said to be linearly dependent if there exist scalars $c_1, c_2, \ldots, c_k$, *not all zero, such that the equation*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$$

holds true.

2. The vectors are said to be linearly independent if the only solution to the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$$

is when all the coefficients $c_1, c_2, \ldots, c_k$ are zero. In other words, there is no nontrivial linear combination of the vectors that equals the zero vector.

The possible spans of vectors depend on their linear dependence or independence.

Example 1.3 *In $\mathbb{R}^2$, there are three different cases:*

Case 1: Linearly Dependent Vectors

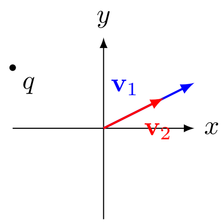
If the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent, there are nonzero numbers c_1 and c_2 in $\mathbb{R}^2$ such that $c_1v_1 + c_2v_2 = 0$.

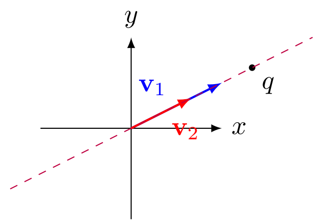
$$c_1v_1 + c_2v_2 = 0 \Leftrightarrow v_2 = -\frac{c_1}{c_2}v_1$$

Thus the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ lie on the same line (are scalar multiples of each other), then the span of these vectors will be the line they lie on. The span will be a one-dimensional subspace.

$$span(v_1, v_2) = \left\{ av_1 + bv_2 \middle| a, b \in \mathbb{R} \right\} = \left\{ av_1 + b(-\frac{c_1}{c_2})v_1 \middle| a, b \in \mathbb{R} \right\} = \left\{ (a - \frac{b\,c_1}{c_2})v_1 \middle| a, b \in \mathbb{R} \right\} = span(v_1)$$

$$span(\mathbf{v}_1, \mathbf{v}_2) = span(v_1) = line\ in\ \mathbb{R}^2$$

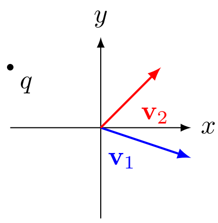




Case 2: Linearly Independent Vectors

If we have two vectors $\mathbf{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} c \\ d \end{bmatrix}$ in $\mathbb{R}^2$ where $\begin{bmatrix} a \\ b \end{bmatrix} \neq \alpha \begin{bmatrix} c \\ d \end{bmatrix}$ for any $\alpha \in \mathbb{R}$, then the span of these vectors will be the entire $\mathbb{R}^2$ space. In other words, any vector in $\mathbb{R}^2$ can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$span(\mathbf{v}_1, \mathbf{v}_2) = \mathbb{R}^2$$

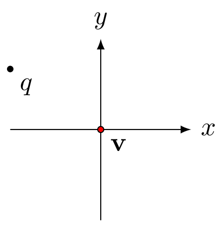


Case 3: Zero Vector

If both of the vectors $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are the zero vector* $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$*, then the span will only consist of the zero vector itself.*

$$span(\mathbf{v}_1, \mathbf{v}_2) = \{\mathbf{0}\}$$

In summary, the possible spans of vectors in $\mathbb{R}^2$ *are the entire* $\mathbb{R}^2$ *space, a line in* $\mathbb{R}^2$*, or just the zero vector depending on the linear dependence or independence of the vectors.*



Example 1.4 *Consider the following set of vectors in* $\mathbb{R}^3$*:*

$$B = \{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \}$$

we prove that they are linearly independent. To show linear independence, we need to demonstrate that the equation $c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ *has only the trivial solution* $c_1 = c_2 = c_3 = 0$*.*

Suppose we have:

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This leads to the following system of equations:

$$\begin{cases} c_1 - c_2 = 0 \\ c_1 + 2c_2 - c_3 = 0 \\ 3c_2 + 2c_3 = 0 \end{cases}$$

Solving this system, we find that $c_1 = c_2 = c_3 = 0$ *is the only solution. Hence, the vectors in* B *are linearly independent.*

2 LU factorization of a matrix A and linear independency of its rows

In this section, we provide a proof demonstrating that if the rows of an $n \times n$ matrix A are dependent, then U, in the LU factorization of the matrix, contains zero rows.

We begin with the equation $A = LU$, where U is obtained through Gaussian elimination and is in row-echelon form. Therefore, if there is a zero row in U, it must be the bottom row of the matrix, then $U_n = 0$. Also, L is an invertible lower triangular matrix. We know that the inverse of L is also a lower triangular matrix which is the product of the row elementary operations that we applied on A to obtain U. The matrix L^{-1} can be represented as follows:

$$L^{-1} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ e_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \dots & 1 \end{bmatrix}$$

Applying L^{-1} on A gives us U.

$$L^{-1}A = U \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & \dots & 0 \\ e_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \dots & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_n \end{bmatrix} = \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \vdots \\ \mathbf{U}_n \end{bmatrix}$$

But since the n-th row of U (U_n) is 0, we have:

$$e_{n1}A_1 + e_{n2}A_2 + \cdots + e_{nn-1}A_{n-1} + A_n = 0$$

This means that the rows of A are dependent if one of the rows in U is zero.

On the other hand, if the rows of A are dependent, there are scalars $c_1, c_2, ..., c_n$ such that the equation $c_1A_1 + c_2A_2 + \cdots + c_{n-1}A_{n-1} + c_nA_n = 0$ holds true. Without loss of generality, let's assume that $c_n \neq 0$. Then we have:

$$\frac{c_1}{c_n}A_1 + \frac{c_2}{c_n}A_2 + ... + \frac{c_{n-1}}{c_n}A_{n-1} + A_n = 0.$$

Now, we apply the following elimination matrices on A:

$$e_1 := \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{c_1}{c_n} & 0 & \dots & 1 \end{bmatrix}, \quad e_2 := \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{c_2}{c_n} & \dots & 1 \end{bmatrix}, \cdots, e_{n-1} := \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Finally, we multiply these elimination matrices with A:

$$e_{n-1}e_{n-2}\cdots e_2e_1A = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{c_1}{c_n} & \frac{c_2}{c_n} & \dots & 1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ \frac{c_1}{c_n}A_1 + \frac{c_2}{c_n}A_2 + ... + \frac{c_{n-1}}{c_n}A_{n-1} + A_n \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ 0 \end{bmatrix}$$

This gives us a matrix where the last row is zero. We can continue applying other elimination matrices to get the LU factorization of A, but the last row will remain zero. Thus, if the rows of A are dependent, then the last row of U in the LU factorization will be zero.

Corollary 2.1 *In the LU factorization of a matrix A, the span of the row vectors of A is equivalent to the span of the nonzero row vectors of U.*

Proof 2.2 *The $A = LU$ implies that every row $A_1, A_2, \cdots, A_n$ is a linear combinations of U rows:*

$$A = LU \quad \Rightarrow \quad \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = L \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ l_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \dots & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix} = \begin{bmatrix} U_1 \\ l_{21}U_1 + U_2 \\ \vdots \\ l_{n1}U_1 + l_{n2}U_2 + \cdots + l_{nn-1}U_{n-1} + U_n \end{bmatrix}$$

Thus:

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix} U_1 \\ l_{21}U_1 + U_2 \\ \vdots \\ l_{n1}U_1 + l_{n2}U_2 + \cdots + l_{nn-1}U_{n-1} + U_n \end{bmatrix}$$

Similarly, since $U = L^{-1}A$ every rows in U is a linear combination of A. Thus:

$$span(A_1, A_2, \cdots A_n) = span(U_1, U_2, \cdots U_n)$$

Therefore, the row spaces of A and U coincide. But we can remove rows that are equal to zero from generator of the $span(U_1, U_2, \cdots U_n)$. That is to say, the set of all linear combinations of the row vectors of A exactly matches the set of all linear combinations of the nonzero row vectors of U.

Corollary 2.3 *In the LU factorization of a matrix A, the number of linearly independent rows in A is equivalent to the number of non-zero rows in U.*

Proof 2.4 *During the LU factorization, Gaussian elimination is applied to the matrix A to obtain the upper triangular matrix U. Any linearly dependent rows in A will be transformed into zero rows in U. Thus, each non-zero row in U corresponds to a linearly independent row in A.*

Consequently, the number of non-zero rows in U equals the number of linearly independent rows in A. This implies that the LU factorization not only gives us information about the structure of the matrix A, but also provides valuable insight into the linear independence of its rows.

Lemma 2.5 *Let $A = LU$ be a composition of $n \times n$ matrix A. All non-zero rows in the upper triangular matrix U are linearly independent.*

Proof 2.6 *Since all zero rows are in the bottom of the matrix U, we can suppose that $U_1, U_2, \ldots, U_k$ are nonzero for $k \leq n$. We will proof that they are independent. Assume the contrary, that there exists a non-trivial linear combination of the rows of U that equals the zero vector. Let $c_1U_1 + c_2U_2 + \ldots + c_kU_k = 0$, where not all c_i are zero. Let i be the smallest index such that $c_i \neq 0$. It means $c_1, c_2, \cdots, c_{i-1}$ are zero. Then:*

$$c_1U_1 + c_2U_2 + \ldots + c_kU_k = 0 \quad \to \quad c_iU_i + c_{i+1}U_{i+1} + \ldots + c_kU_k = 0$$

Thus U_i is a linear combination of $U_{i+1}, \ldots, U_k$ as follows:

$$U_i = \frac{c_{i+1}}{c_i}U_{i+1} + \ldots + \frac{c_k}{c_i}U_k.$$

But U_i has a pivot entry that is nonzero, and all entry below the pivot entry are zero. So $U_{i+1}, \ldots, U_k$ the entry in the same position is zero. This results in a contradiction since the left-hand side sum cannot equal the U_i vector, thus all non-zero rows in U must be linearly independent.

Example 2.7 *We're given a matrix A which is an upper triangular matrix.*

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 & 0 \\ 0 & 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The three non-zero rows in the matrix are:

$$\mathit{Row1} = \begin{bmatrix} 1 & 3 & -1 & 2 & 0 \end{bmatrix}, \quad \mathit{Row2} = \begin{bmatrix} 0 & 0 & 2 & 1 & -1 \end{bmatrix}, \quad \mathit{Row3} = \begin{bmatrix} 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

We want to prove that these rows are linearly independent. This means that there are no coefficients c_1, c_2, and c_3 (not all zero) such that $c_1 \cdot \mathit{Row1} + c_2 \cdot \mathit{Row2} + c_3 \cdot \mathit{Row3} = \mathbf{0}$. In other words, there's no way to add or subtract multiples of these rows to make the zero vector.

Now, let's assume the opposite. Let's say there are coefficients c_1, c_2, and c_3 (not all zero) such that $c_1 \cdot \mathit{Row1} + c_2 \cdot \mathit{Row2} + c_3 \cdot \mathit{Row3} = 0$.

Consider the scenario where c_1 is the smallest nonzero coefficient. We have the equation:

$$c_1 \cdot \mathit{Row1} + c_2 \cdot \mathit{Row2} + c_3 \cdot \mathit{Row3} = 0$$

Isolating Row1, we get:

$$\mathit{Row1} = -\frac{c_2}{c_1} \cdot \mathit{Row2} - \frac{c_3}{c_1} \cdot \mathit{Row3}$$

The right-hand side of this equation is a linear combination of Row2 and Row3. However, if we examine the individual elements in Row1, Row2, and Row3, it's clear that there's no way to combine Row2 and Row3 to yield Row1.

In particular, consider the first elements of each vector. The first element of Row1 is 1, whereas the first elements of Row2 and Row3 are both 0. Thus, no linear combination of Row2 and Row3 can yield a vector whose first element is 1.

This leads to a contradiction, and so we conclude that if c_1 is the smallest nonzero coefficient, our original assumption that the rows were not linearly independent was false.

Suppose that 2 be the smallest index where c_i is not equal to zero. This gives us:

$$c_2 \cdot \mathit{Row2} + c_3 \cdot \mathit{Row3} = 0$$

Which simplifies to:

$$\mathit{Row2} = -\left(\frac{c_3}{c_2}\right) \cdot \mathit{Row3}$$

This equation suggests that Row2 is a scalar multiple of Row3. However, when we compare the actual vectors, we can see that this is not true. The third element of Row2 is 2, while the third element of $-(\frac{c_3}{c_2}) \cdot \mathit{Row3}$ is 0, which is a contradiction.

If c_3 is nonzero (i.e., 3 is the smallest index for which c_i is nonzero), then the equation would look like this:

$$c_3 \mathit{Row3} = 0$$

Again, this would mean that Row3 equals the zero vector, which is clearly not true. Hence, we conclude that Row1, Row2, and Row3 are indeed linearly independent.

3 Bases of Cartesian Product $\mathbb{R}^n$

In the Cartesian product $\mathbb{R}^n$, the bases can be defined in terms of span and linear independence as follows:

1. **Span**: A set of vectors $\{v_1, v_2, ..., v_k\}$ is said to span $\mathbb{R}^n$ if every vector in $\mathbb{R}^n$ can be expressed as a linear combination of those vectors. In other words, for any vector $\mathbf{x} = (x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$, there exist scalars $c_1, c_2, ..., c_k$ such that $\mathbf{x} = c_1v_1 + c_2v_2 + ... + c_kv_k$.

2. **Linear Independence**: A set of vectors $\{v_1, v_2, ..., v_k\}$ is said to be linearly independent in $\mathbb{R}^n$ if no vector in the set can be expressed as a linear combination of the others. This means that for any scalars $c_1, c_2, ..., c_k$, the equation $c_1v_1 + c_2v_2 + ... + c_kv_k = 0$ has only the trivial solution $c_1 = c_2 = ... = c_k = 0$.

In the context of linear independence, removing any vector from the basis would reduce its ability to span the entire space. This is why the basis vectors are sometimes likened to a "building kit" for the vector space: Just like how you can't build a specific model without all the necessary building blocks, you can't cover the entire vector space without all the basis vectors.

Therefore, a basis is indeed the "minimal" set of vectors that can describe the entire space. This principle applies not only to finite dimensional spaces like $\mathbb{R}^n$, but to any vector space in general.

4 Efficient Computation of Basis Using LU Factorization

In the field of linear algebra, given a matrix A, if one wishes to compute a basis for the space spanned by the rows of A, an effective technique is to perform an LU factorization on A. The LU factorization decomposes $A = LU$ into a product of a lower triangular matrix L and an upper triangular matrix U.

In the context of vector spaces, a basis is defined as a set of linearly independent vectors that span the entire space. The nonzero rows of the upper triangular matrix U in the LU decomposition of A are linearly independent (Lemma 2.5), and they span the same space as the rows of A (Corollary 2.1). Consequently, these rows of U constitute a basis for the row space of A.

This understanding offers a practical approach for calculating the basis of the row space of a matrix. Given a matrix A, one can perform an LU factorization to find U and then use the nonzero rows of U as the basis. This procedure can be more efficient than alternative methods such as applying Gaussian elimination on A and selecting the nonzero rows of the resulting row-echelon form.

Example 4.1 *Consider the following 3×3 matrix A:*

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 6 & 7 \\ 4 & 8 & 4 \end{bmatrix}$$

We can calculate the LU factorization of this matrix, which results in the following L and U matrices:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

The first and second rows of matrix U are nonzero. The third row in U is zero, indicates that there is a linear dependence in the corresponding rows of A.

We can now see that the first and second row of U can be used as the basis for the row space of A. These rows are linearly independent, and every row in A is a linear combination of these two rows.

Hence, the first and second row vectors of U indeed form a basis for the row space of A.

Example 4.2 *Consider a 4×4 matrix A defined as:*

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & 0 & 1 \\ 1 & 2 & -1 & 3 \\ 3 & 6 & 1 & 4 \end{bmatrix}$$

We can compute the LU factorization of this matrix, yielding:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 2 & 2/3 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We can see that the first, second, third and fourth rows of matrix U are $(1, 2, -1, 0)$, $(0, 0, 2, 1)$, $(0, 0, 0, 3)$ and $(0, 0, 0, 0)$ respectively. The zero row in U indicates that there is linear dependence in the corresponding rows of A. In this case, the fourth row of A is a linear combination of the first, second and third rows.

We can use the first, second, and third row of U as the basis for the row space of A. These rows are clearly linearly independent (no combination of any two rows gives the third row), and every row in A is a linear combination of these three rows. So, these three vectors indeed form a basis for the row space of A.

This example demonstrates the practical utility of the LU factorization in finding a basis for the row space of a matrix.