Vectors

Friday 28.07.2023 (Room U3)

1 Overview

Definition 1.1 *A vector* $\mathbf{v}$ *over the real numbers* $\mathbb{R}$ *can be represented as:*

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \langle v_1, v_2, \ldots, v_n \rangle \in \mathbb{R}^n$$

where $\mathbf{v}$ *is a column vector with* n *components* $v_1, v_2, \ldots, v_n$*. Each component* v_i *belongs to the set of real numbers* $\mathbb{R}$*.*

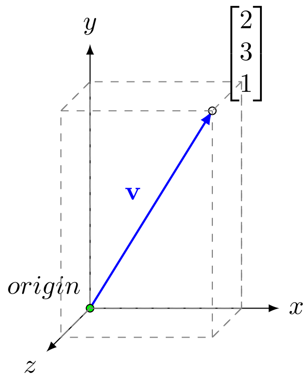


Figure 1: Three-Dimensional Vector Representation

Definition 1.2 *The set* $\mathbb{R}^n$ *represents the* n*-dimensional Cartesian product (Euclidean space) over the real numbers. It consists of all possible vectors of length* n *whose components are real numbers.*

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle| x_1, x_2, \ldots, x_n \in \mathbb{R} \right\} = \underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{n}$$

Each vector in $\mathbb{R}^n$ *can be represented as an ordered collection of* n *real numbers, where* $x_1, x_2, \ldots, x_n$ *are the components of the vector.*

Example 1.3 *The* $\mathbb{R}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1, x_2 \in \mathbb{R} \right\}$ *represents all points in a 2-dimensional Cartesian coordinate system. The first component of* $\mathbf{v}$ *represents the* x*-coordinate, and the second component represents the* y*-coordinate.*

Definition 1.4 *The origin is a refers to a specific point that serves as a starting point in a coordinate system. In a two-dimensional Cartesian coordinate system, the origin is typically denoted* *as* $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. *In a three-dimensional Cartesian coordinate system, the origin is denoted as* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. *It represents the point of intersection of the coordinate axes.*

2 Vector Operations

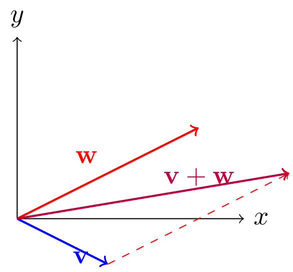
Vector operations play a crucial role in understanding and manipulating vectors within vector spaces. We define the addition of vectors, scalar multiplication, and linear combination of vectors as follows:

2.1 Addition of Vectors

Definition 2.1 *Given vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *in a vector space* V *over the real numbers, the sum of* $\mathbf{v}$ *and* $\mathbf{w}$*, denoted by* $\mathbf{v}+\mathbf{w}$*, is a vector obtained by adding the corresponding components of* $\mathbf{v}$ *and* $\mathbf{w}$*:*

$$\mathbf{v}+\mathbf{w} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1+w_1 \\ v_2+w_2 \\ \vdots \\ v_n+w_n \end{bmatrix}$$

Example 2.2 *Consider the vectors* $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ *and* $\mathbf{w} = \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}$. *Their sum is given by* $\mathbf{v}+\mathbf{w} = \begin{bmatrix} 2+4 \\ -1+2 \\ 3+(-5) \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ -2 \end{bmatrix}$.



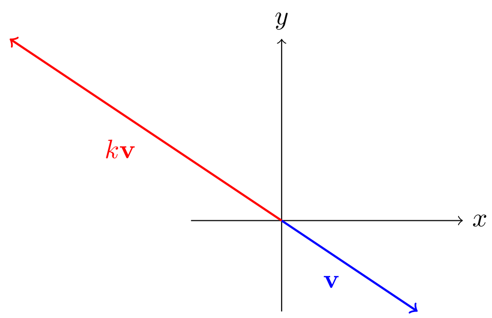
The addition of vectors allows us to combine their respective components, resulting in a new vector that represents their combined effect.

2.2 Scalar Multiplication

Definition 2.3 *Let* $\mathbf{v}$ *be a vector in a vector space* V *over the real numbers, and let* k *be a real number. The scalar multiplication of* k *and* $\mathbf{v}$*, denoted by* $k\mathbf{v}$*, yields a vector obtained by multiplying each component of* $\mathbf{v}$ *by* k*:*

$$
k\mathbf{v} = k \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} kv_1 \\ kv_2 \\ \vdots \\ kv_n \end{bmatrix}
$$

Example 2.4 *Consider the vector* $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ *and* $k = -2$. *The scalar multiplication* $k\mathbf{v}$ *is given by* $-2\mathbf{v} = \begin{bmatrix} -2(3) \\ -2(-2) \\ -2(1) \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \\ -2 \end{bmatrix}$.



Scalar multiplication allows us to stretch or shrink vectors by scaling their components, resulting in a vector that maintains the same direction but varies in magnitude.

3 Magnitude of a Vector and its Properties

For a vector $\mathbf{v}$ in $\mathbb{R}^n$, its magnitude $|\mathbf{v}|$ is calculated as:

$$
|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}
$$

where $v_1, v_2, \ldots, v_n$ are the components of $\mathbf{v}$. The rules or properties related to vector magnitudes include:

1. **Non-Negativity:** The magnitude of a vector is always non-negative, i.e., $|\mathbf{v}| \geq 0$.
2. **Zero Magnitude:** The magnitude of a vector is zero if and only if all its components are zero. In other words, $|\mathbf{v}| = 0$ if and only if $v_1 = v_2 = \ldots = v_n = 0$.
3. **Scalar Multiplication:** If $\mathbf{v}$ is a vector and c is a scalar, then the magnitude of the scalar multiple $c\mathbf{v}$ is given by $|c\mathbf{v}| = |c||\mathbf{v}|$. In other words, multiplying a vector by a scalar multiplies its magnitude by the absolute value of that scalar.
4. **Triangle Inequality:** For any two vectors $\mathbf{u}$ and $\mathbf{v}$, the magnitude of their sum is less than or equal to the sum of their magnitudes, i.e., $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$.
5. **Magnitude of the Sum:** For any two vectors $\mathbf{u}$ and $\mathbf{v}$, we can calculate the magnitude of their sum as follows: $|\mathbf{u} + \mathbf{v}| = \sqrt{|\mathbf{u}|^2 + |\mathbf{v}|^2 + 2|\mathbf{u}||\mathbf{v}|\cos(\theta)}$. See here for the proof.

To further explore vector analysis, we can delve into the study of the dot product and cross product operations. These mathematical operations offer powerful tools for calculating the angle between two vectors and gaining deeper insights into their properties and relationships.

4 Dot (Scalar) Product of Vectors in $\mathbb{R}^n$

In this section, we will explore the concept of the dot product of vectors in $\mathbb{R}^n$. The dot product is a fundamental operation that allows us to measure the similarity between vectors, determine angles between vectors, and solve various geometric and algebraic problems. We will discuss its definition, properties, and applications through interesting examples and figures. Understanding the dot product of vectors is essential for developing a strong foundation in linear algebra and related disciplines. Some common applications include:

- Determining whether two vectors are orthogonal or parallel.
- Computing the projection of a vector onto another vector.
- Finding the angle between two vectors.

4.1 Definition

Given two vectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$ in $\mathbb{R}^n$, the dot product, denoted as $\mathbf{v} \cdot \mathbf{w}$, is defined as

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n.$$

4.2 Geometric Interpretation

The dot product can be interpreted geometrically using the concept of angles between vectors. For two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$, the dot product $\mathbf{v} \cdot \mathbf{w}$ is given by

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos(\theta),$$

where $|\mathbf{v}|$ and $|\mathbf{w}|$ are the magnitudes of $\mathbf{v}$ and $\mathbf{w}$, respectively, and θ is the angle between the two vectors.

The dot product is a concept that is commonly presented as both an algebraic and a geometric operation. However, the connection between these two perspectives may not be immediately evident. It is natural to wonder how these two interpretations are equivalent and why such a relationship exists. If you're interested in exploring the proof of their equivalence and gaining a deeper understanding of why this relationship makes sense, you can find more information through the following link: Proof of Equivalence between Algebraic and Geometric Interpretations of the Dot Product. This resource will provide a comprehensive explanation, allowing you to appreciate the intuitive connection between the algebraic and geometric aspects of the dot product.

4.3 Properties of Dot Product

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^n$. The dot product of $\mathbf{u}$ and $\mathbf{v}$, denoted as $\mathbf{u} \cdot \mathbf{v}$, possesses the following properties:

1. **Commutative Property:** $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. **Distributive Property:** $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
3. **Scalar Multiplication:** $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$
4. **Orthogonal Vectors:** If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors, i.e., $\mathbf{u} \perp \mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = 0$
5. **Length Preservation:** The dot product of a vector with itself gives the square of its magnitude:

$$\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$$

These properties are fundamental in vector algebra and have applications in various fields of mathematics, physics, and engineering.

Example 4.1 ***Computing the Dot Product***

Consider the vectors $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$. *We will compute the dot product and the angle between these vectors. The dot product of* $\mathbf{u}$ *and* $\mathbf{v}$*, denoted as* $\mathbf{u} \cdot \mathbf{v}$*, is given by the formula:*

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Substituting the values, we have:

$$\mathbf{u} \cdot \mathbf{v} = (2)(0) + (-1)(4) + (3)(1) = -4 + 3 = -1.$$

Therefore, $\mathbf{u} \cdot \mathbf{v} = -1$. *To compute the angle between the vectors* $\mathbf{u}$ *and* $\mathbf{v}$*, we can use the dot product formula:*

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|},$$

where $|\mathbf{u}|$ *and* $|\mathbf{v}|$ *represent the magnitudes of the vectors. The magnitude of a vector is calculated as:*

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

Substituting the values, we have:

$$|\mathbf{u}| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}.$$

Similarly, the magnitude of $\mathbf{v}$ *is:*

$$|\mathbf{v}| = \sqrt{0^2 + 4^2 + 1^2} = \sqrt{17}.$$

Substituting these values into the angle formula, we find:

$$\cos(\theta) = \frac{-1}{\sqrt{14} \cdot \sqrt{17}}.$$

Simplifying the expression, we have:

$$\cos(\theta) = \frac{-1}{\sqrt{238}}.$$

Taking the inverse cosine, we find:

$$\theta = \arccos\left(\frac{-1}{\sqrt{238}}\right).$$

Using a calculator we can find the approximate value of θ.

4.4 Magnitude of Sum

To compute the magnitude of the sum of two vectors $\mathbf{u}$ and $\mathbf{v}$ using their individual magnitudes, denoted as $|\mathbf{u}|$ and $|\mathbf{v}|$, we can use the following formula:

By knowing the magnitudes of the individual vectors and the angle between them, we can use this formula to find the magnitude of their sum:

$$|\mathbf{u} + \mathbf{v}| = \sqrt{|\mathbf{u}|^2 + 2|\mathbf{u}||\mathbf{v}|\cos(\theta) + |\mathbf{v}|^2}$$

where θ is the angle between vectors $\mathbf{u}$ and $\mathbf{v}$. To prove that we can start with the following formula:

$$|\mathbf{u} + \mathbf{v}|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$$

Expanding the dot product:

$$|\mathbf{u} + \mathbf{v}|^2 = \mathbf{u} \cdot \mathbf{u} + 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v}$$

Using the properties of the magnitude, $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$ and $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$, we can substitute them in the equation:

$$|\mathbf{u} + \mathbf{v}| = \sqrt{|\mathbf{u}|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2}$$

Next, we can use the cosine rule for the dot product:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos(\theta)$$

where θ is the angle between vectors $\mathbf{u}$ and $\mathbf{v}$. Substituting this in the equation:

$$|\mathbf{u} + \mathbf{v}| = \sqrt{|\mathbf{u}|^2 + 2|\mathbf{u}||\mathbf{v}|\cos(\theta) + |\mathbf{v}|^2}$$

which proves the desired result.

Definition 4.2 ***Normal Vector***

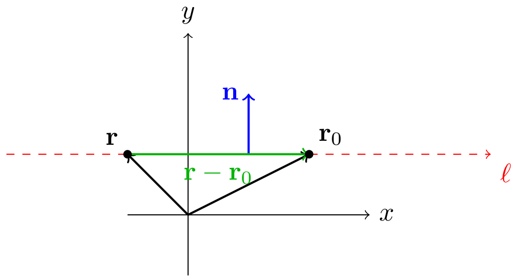
A normal vector is a vector that is orthogonal to a given surface or another vector.

In both two-dimensional space $\mathbb{R}^2$ and three-dimensional space $\mathbb{R}^3$, consider a vector $\mathbf{r}_0$ and a vector $\mathbf{n}$. What is the set of all vectors $\mathbf{r}$ in both $\mathbb{R}^2$ and $\mathbb{R}^3$ such that $\mathbf{n}$ is orthogonal to the vector $\mathbf{r} - \mathbf{r}_0$? Can you provide the equation in coordinate form that represents this set for both $\mathbb{R}^2$ and $\mathbb{R}^3$?

Since $\mathbf{n}$ is orthogonal to $\mathbf{r} - \mathbf{r}_0$ we have $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$. Expanding this equation, we have

$$(x - x_0)n_1 + (y - y_0)n_2 = 0,$$

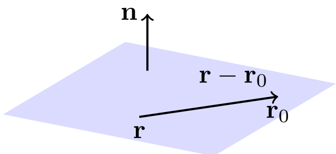
where $\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{r}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ are the coordinate vectors of $\mathbf{r}$ and $\mathbf{r}_0$ respectively, and $\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$ is the coordinate vector of $\mathbf{n}$. Therefore, in $\mathbb{R}^2$, the set of all vectors $\mathbf{r}$ such that $\mathbf{n}$ is orthogonal to $\mathbf{r} - \mathbf{r}_0$ forms a line.



The same strategy works in $\mathbb{R}^3$ and $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$. Expanding this equation, we have

$$(x - x_0)n_1 + (y - y_0)n_2 + (z - z_0)n_3 = 0,$$

where $\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathbf{r}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$, and $\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$. Therefore, in $\mathbb{R}^3$, the set of all vectors $\mathbf{r}$ such that $\mathbf{n}$ is orthogonal to $\mathbf{r} - \mathbf{r}_0$ forms a plane.



5 Cross Product of Vectors in $\mathbb{R}^3$

The cross product, also known as the vector product, is a binary operation between two vectors in three-dimensional space ($\mathbb{R}^3$). It produces a new vector that is orthogonal to both of the original vectors and captures important geometric properties.

Given two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ in $\mathbb{R}^3$, their cross product $\mathbf{u} \times \mathbf{v}$ is defined using the matrix determinant as:

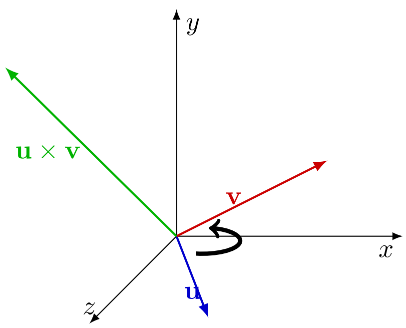
$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \mathbf{i}(u_2v_3 - u_3v_2) - \mathbf{j}(u_1v_3 - u_3v_1) + \mathbf{k}(u_1v_2 - u_2v_1)$$

where $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Expanding this determinant, we have:

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

The resulting vector, $\mathbf{u} \times \mathbf{v}$, is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, lying in a plane orthogonal to the plane containing the original vectors. Its direction is determined by the right-hand rule: if you curl the fingers of your right hand from $\mathbf{u}$ towards $\mathbf{v}$, the extended thumb points in the direction of $\mathbf{u} \times \mathbf{v}$.



The cross product has several important properties:

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ (Anticommutative property)
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ (Distributive property)
3. $(k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v}) = k(\mathbf{u} \times \mathbf{v})$ (Scalar multiplication property)
4. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ (Triple scalar product)

Example 5.1 *Let* $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$. *We can find their cross product as follows:*

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 4 & 2 & 0 \end{vmatrix} = \begin{bmatrix} (-1 \times 0) - (3 \times 2) \\ (3 \times 4) - (2 \times 0) \\ (2 \times 2) - (-1 \times 4) \end{bmatrix} = \begin{bmatrix} -6 \\ 12 \\ 8 \end{bmatrix}$$

Example 5.2 *Consider the vectors* $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. *Their cross product is:*

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (0 \times 0) - (0 \times 1) \\ (0 \times 0) - (1 \times 0) \\ (1 \times 1) - (0 \times 0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The magnitude of the cross product, denoted $|\mathbf{u} \times \mathbf{v}|$, can be calculated using the Euclidean norm:

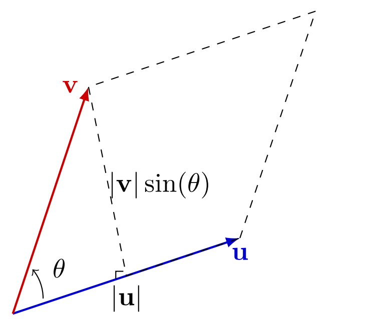
$$|\mathbf{u} \times \mathbf{v}| = \sqrt{(u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2}$$

Also, the magnitude can be computed using the magnitudes of the original vectors and the sine of the angle θ between them:

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin(\theta)$$

where $|\mathbf{u}|$ represents the magnitude of vector $\mathbf{u}$, $|\mathbf{v}|$ represents the magnitude of vector $\mathbf{v}$, and θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

The magnitude of the cross product represents the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.



Area $= |\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| \cdot |\mathbf{v}| \cdot \sin(\theta)$

5.1 Linear Combination of Vectors

Definition 5.3 *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$, and let $a_1, a_2, \ldots, a_k$ be real numbers. The linear combination of these vectors with coefficients $a_1, a_2, \ldots, a_k$ is defined as the vector:*

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_k\mathbf{v}_k = a_1 \begin{bmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{bmatrix} + a_2 \begin{bmatrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \end{bmatrix} + \ldots + a_k \begin{bmatrix} v_{k1} \\ v_{12} \\ \vdots \\ v_{kn} \end{bmatrix} = \begin{bmatrix} a_1v_{11} + a_2v_{21} + \ldots + a_kv_{k1} \\ a_1v_{12} + a_2v_{22} + \ldots + a_kv_{k2} \\ \vdots \\ a_1v_{1n} + a_2v_{2n} + \ldots + a_kv_{kn} \end{bmatrix}$$

where v_{ij} represents the j-th component of the vector $\mathbf{v}_i$.

Example 5.4 *Consider the vectors* $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$, *and* $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ *with coefficients $a_1 = 2$, $a_2 = -1$, and $a_3 = 3$. The linear combination $2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3$ is given by:*

$$2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3 = 2\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \\ 15 \end{bmatrix} = \begin{bmatrix} -7 \\ 7 \\ 9 \end{bmatrix}$$

Linear combinations allow us to express vectors as weighted combinations of other vectors, providing a versatile tool for representing vector relationships and generating new vectors within vector spaces. In Euclidean space $\mathbb{R}^2$, every point can be represented as a linear combination of only two vectors.

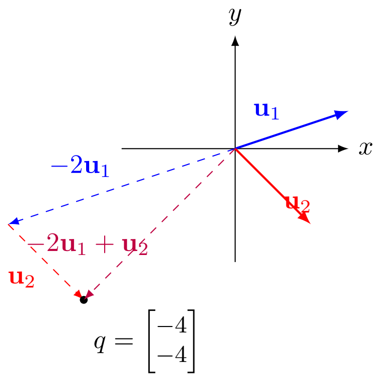
Example 5.5 *In $\mathbb{R}^2$, consider the column vectors $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$. Let's take another point $\mathbf{q} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$. To express $\mathbf{q}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we need to find x, y as follows:*

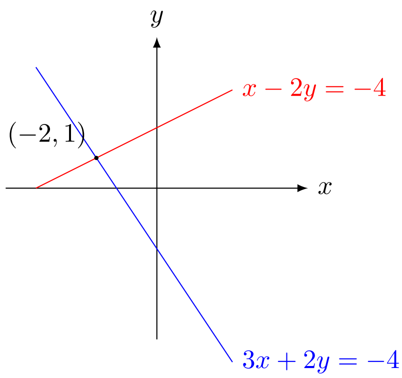
$$x \begin{bmatrix} 3 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$$

The process of solving this problem involving vectors can be reformulated as tackling a system of equations or a problem involving a matrix.

$$x \begin{bmatrix} 3 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} 3x + 2y & = -4 \\ x - 2y & = -4 \end{cases} \quad \Leftrightarrow \quad \begin{bmatrix} 3 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$$

The $\mathbf{q}$ can be express by scaling $\mathbf{u}_1$ by -2 and $\mathbf{v}_2$ by 1, and then adding them together.





Furthermore, later on, we will discover that every point in $\mathbb{R}^2$ can be expressed as a linear combination of the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$. Our exploration also will focus on understanding the specific properties of $\mathbf{u}_1$ and $\mathbf{u}_2$ that enable us to represent any point in $\mathbb{R}^2$ using these vectors.

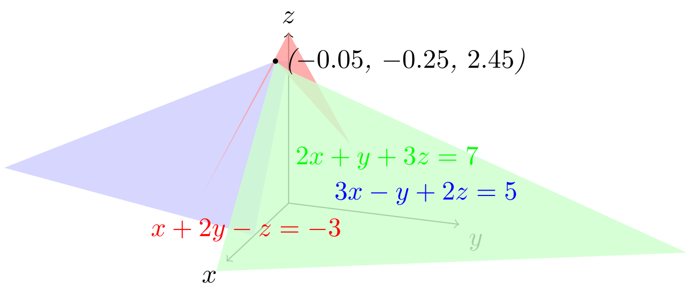
Example 5.6 *We want to express the vector $\begin{bmatrix} 5 \\ -3 \\ 7 \end{bmatrix}$ in $\mathbb{R}^3$ as a linear combination of $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$. Indeed we need to find x, y, z such that:*

$$x \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + z \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 7 \end{bmatrix}$$

But

$$x \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + z \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 7 \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} 3x - y + 2z & = 5 \\ x + 2y - z & = -3 \\ 2x + y + 3z & = 7 \end{cases} \quad \Leftrightarrow \quad \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 7 \end{bmatrix}$$

The system of equations has a solution where the planes intersect at the point $(-0.05, -0.25, 2.45)$*. Indeed, the solution to the given system of equations is* $x = -0.05$*,* $y = -0.25$*, and* $z = 2.45$*. As the dimension of the vector increases, it becomes increasingly difficult to visualise the solution geometrically.*



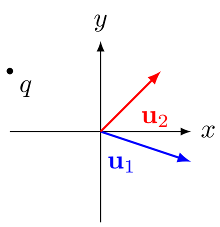
A system of equations can have various types of solutions, depending on the nature of the equations and the variables involved. Here are some possible solution scenarios for a system of equations:

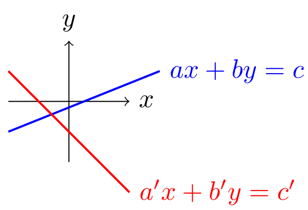
- Unique Solution: The system has a single solution, meaning there is a unique set of values for the variables that satisfies all the equations.
- No Solution: The system has no solution, indicating that there are no values for the variables that simultaneously satisfy all the equations.
- Infinite Solutions: The system has infinitely many solutions, implying that there are multiple sets of values for the variables that satisfy the equations. In this case, the equations are dependent or redundant, and the solution set can be expressed in terms of parameters.

In $\mathbb{R}^2$, a system of equations can have three possible types of solutions: a unique solution, no solution, or infinitely many solutions.

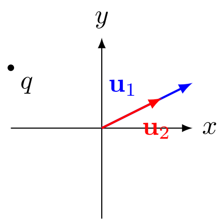
Unique Solution: In this case, the system of equations has a single point of intersection. Geometrically, the lines representing the equations intersect at a specific point in the Cartesian plane. No Solution: In this case, the system of equations has no common point of intersection. Geometrically, the lines representing the equations are parallel and do not intersect. Infinitely Many Solutions: In this case, the system of equations has an infinite number of common points of intersection. Geometrically, the lines representing the equations are coincident (overlap) and intersect at every point along the lines.

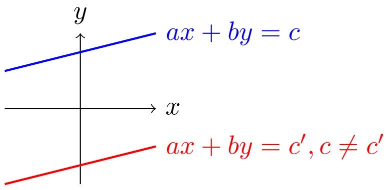
1. Unique Solution:





2. No Solution:





3. Infinite Solutions: