A = LU

Friday 04.08.2023 (Room U3)

1 Overview

LU decomposition, also known as LU factorization, is a method used to factorize a square matrix into the product of two matrices: an upper triangular matrix (U) and a lower triangular matrix (L). This decomposition can be mathematically represented as:

$$A = LU.$$

The decomposition is typically obtained using Gaussian elimination. The goal is to transform the original matrix into an upper triangular form (U), while simultaneously creating a lower triangular matrix (L) that captures the elimination operations performed.

The process of LU decomposition involves performing row operations to eliminate the variables below the main diagonal and obtain an upper triangular matrix (U) while simultaneously constructing a lower triangular matrix (L) that stores the multipliers used in the elimination process.

The LU decomposition has several applications, including solving systems of linear equations, computing the determinant of a matrix, and finding the inverse of a matrix. By decomposing a matrix into two triangular matrices, the process of solving linear systems and performing other matrix operations becomes more efficient.

Lemma 1.1 *Let A and B denote matrices. We present two key properties related to lower and upper triangular matrices.*

1. ***Product of Lower (Upper) Triangular Matrices:*** *If A and B are both lower (upper) triangular matrices, then the product AB is also a lower (upper) triangular matrix.*
2. ***Invertibility of Lower (Upper) Triangular Matrices:*** *If A is an $n \times n$ lower (upper) triangular matrix, then A is invertible if and only if every main diagonal entry is nonzero. In this case, the inverse A^{-1} is also a lower (upper) triangular matrix.*

Example 1.2 *Let's consider the following matrix A:*

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 4 & 4 & 6 \\ -2 & 3 & 1 \end{bmatrix}$$

We use Gaussian elimination to find L and U such that $A = LU$. Every elementary row operation can by done by multiplication of an invertible elimination matrix as follows:

1. *Subtract 2 times the first row from the second row, by multiplying the following matrix e_1:*

$$e_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 4 & 4 & 6 \\ -2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & -2 \\ -2 & 3 & 1 \end{bmatrix}$$

2. Add the first row to the second row, using the e_2 matrix

$$e_2(e_1A) = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}(e_1A) = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & 4\\0 & 2 & -2\\-2 & 3 & 1\end{bmatrix} = \begin{bmatrix}2 & 1 & 4\\0 & 2 & -2\\0 & 4 & 5\end{bmatrix}$$

3. Subtract 2 times the second row from the third row using e_3

$$e_3(e_2e_1A) = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -2 & 1\end{bmatrix}(e_2e_1A) = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -2 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & 4\\0 & 2 & -2\\0 & 4 & 5\end{bmatrix} = \begin{bmatrix}2 & 1 & 4\\0 & 2 & -2\\0 & 0 & 9\end{bmatrix}$$

Therefore

$$(e_3e_2e_1)A = \begin{bmatrix}2 & 1 & 4\\0 & 2 & -2\\0 & 0 & 9\end{bmatrix} \quad \Rightarrow \quad A = (e_3e_2e_1)^{-1}\begin{bmatrix}2 & 1 & 4\\0 & 2 & -2\\0 & 0 & 9\end{bmatrix}.$$

Set $U := \begin{bmatrix}2 & 1 & 4\\0 & 2 & -2\\0 & 0 & 9\end{bmatrix}$ *and* $L := (e_3e_2e_1)^{-1}$. *Let us calculate* L :

$$L = (e_3e_2e_1)^{-1} = e_1^{-1}e_2^{-1}e_3^{-1} = \begin{bmatrix}1 & 0 & 0\\2 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 2 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\2 & 1 & 0\\-1 & 2 & 1\end{bmatrix}$$

Now we can verify the decomposition by multiplying L *and* U*:*

$$LU = \begin{bmatrix}1 & 0 & 0\\2 & 1 & 0\\-1 & 2 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & 4\\0 & 2 & -2\\0 & 0 & 9\end{bmatrix} = \begin{bmatrix}2 & 1 & 4\\4 & 4 & 6\\-2 & 3 & 1\end{bmatrix} = A$$

As you can see, the product of L *and* U *gives us the original matrix* A*, confirming the correctness of the* LU *decomposition.*

2 LU decomposition in solving systems of linear equations

LU decomposition is useful in solving systems of linear equations, as it allows us to rewrite the system $AX = B$ as $LUX = B$, where L and U are triangular matrices. By solving two simpler triangular systems, $LY = B$ (forward substitution) and $UX = Y$ (back substitution), we can obtain the solution vector X for the system.

To solve a linear system of equations using LU decomposition, we can follow these steps:

1. Perform LU Decomposition: Convert the system into matrix form ($AX = B$) and decompose matrix A into the product of lower triangular matrix L and upper triangular matrix U.
2. Solve $LY = B$: Solve the lower triangular system $LY = B$ using forward substitution to find the values of Y.
3. Solve $UX = Y$: Solve the upper triangular system $UX = Y$ using back substitution to find the values of X.

Example 2.1 *Let's consider the following system of equations:*

$$\begin{cases} 2x + y + 4z = 12 \\ 4x + 4y + 6z = 26 \\ -2x + 3y + z = 1 \end{cases}$$

1. LU Decomposition: We start by decomposing the coefficient matrix $A = \begin{bmatrix} 2 & 1 & 4 \\ 4 & 4 & 6 \\ -2 & 3 & 1 \end{bmatrix}$ *into L and U matrices that are calculated from previous example:*

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & -2 \\ 0 & 0 & 9 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & -2 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 \\ 4 & 4 & 6 \\ -2 & 3 & 1 \end{bmatrix}$$

2. Solve $LY = B$: We substitute matrix B (the constants from the system) into $LY = B$ and solve for Y using forward substitution:

$$LY = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 26 \\ 1 \end{bmatrix}$$

Solving this lower triangular system, we find:

$$Y = \begin{bmatrix} 12 \\ 2 \\ 9 \end{bmatrix}$$

3. Solve $UX = Y$: We substitute the values of Y into $UX = Y$ and solve for X using back substitution:

$$UX = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & -2 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 2 \\ 9 \end{bmatrix}$$

Solving this upper triangular system, we find:

$$X = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

So, the solution to the given system of equations using LU decomposition is $x = 3$, $y = 2$, $z = 1$.

3 PLU : Incorporating Permutations into LU Decomposition (Partial Pivoting)

In the two previous examples, we made use of elimination matrices that were specifically lower triangular matrices. Consequently, both the product and the inverse of these matrices retain this lower triangular form, hence L (the lower triangular matrix in an LUdecomposition) was always lower triangular. This approach is effective and straightforward in many cases, especially when we are dealing with systems that do not require any row swapping.

However, sometimes during Gaussian elimination, we find the need to swap rows. The corresponding elimination matrix for such row swapping, or permutations, is no longer a lower triangular matrix.

In these situations, we don't completely abandon the LU decomposition. Instead, we modify our approach slightly to include these permutations. To achieve this, we introduce a permutation matrix P. This matrix is used to perform the necessary row swaps on our initial matrix, A. Then PA is ready to be decomposed to LU.

The algorithm then changes to perform the LU decomposition not on the matrix A, but on the product of P and A, denoted by PA. So, instead of directly decomposing A into L and U, we are decomposing PA into L and U. This approach is commonly referred to as the PLU decomposition, and it incorporates the necessary row permutations directly into the process.

In the PLU decomposition, P represents the necessary row permutations, L remains as the lower triangular matrix (with ones on the diagonal), and U is the upper triangular matrix. The benefit of the PLU decomposition is that it maintains the lower triangular form of L while allowing for the necessary row swaps, making it a more versatile approach when handling a wider range of systems of linear equations. This strategy enables us to tackle more complex problems using the same foundational concepts of LU decomposition.

3.1 Permutation Matrices

Permutation matrices, as used in PLU decomposition, have important properties that make them ideal for their role in this decomposition. A permutation matrix is an orthogonal matrix, which means that it is invertible and its inverse is its transpose. Mathematically, this can be stated as $P^T = P^{-1}$.

A permutation matrix P is formed by a series of single swapping elimination matrices e_i for $i = 1, 2, \cdots n$.

$$P = e_n e_{n-1} \cdots e_1 \quad \Rightarrow \quad P^{-1} = (e_n e_{n-1} \cdots e_1)^{-1}$$

But each swapping elimination matrix is its own inverse and orthogonal, as performing the same row swap twice will revert to the original matrix. Thus, P is a product of these invertible matrices and is therefore also invertible.

$$P^{-1} = (e_n e_{n-1} \cdots e_1)^{-1} = e_1^{-1} e_2^{-1} \cdots e_n^{-1} = e_1^T e_2^T \cdots e_n^T = (e_n e_{n-1} \cdots e_1)^T \quad \Rightarrow \quad P^{-1} = P^T$$

Theorem 3.1 *Suppose an $m \times n$ matrix A is carried to a row-echelon matrix U via the Gaussian algorithm. Let $P_1, P_2, \ldots, P_s$ be the elementary matrices corresponding (in order) to the row interchanges used, and write $P = P_s \cdots P_2 P_1$. (If no interchanges are used, take $P = I_m$, where I_m is the $m \times m$ identity matrix.) Then, we have the following properties:*

1. *PA is the matrix obtained from A by performing these interchanges (in order) to A.*
2. *PA has an LU-factorization.*

Example 3.2 *We will solve the following system of equations using PLU (Permutation, Lower, Upper) Decomposition:*

$$\begin{cases} 2z + w = 4 \\ 2x + y + 3z - w = -4 \\ 4y - z + 3w = 1 \\ 4x + 2y + 6z = -4 \end{cases}$$

The coefficient matrix A for this system is:

$$A = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 2 & 1 & 3 & -1 \\ 0 & 4 & -1 & 3 \\ 4 & 2 & 6 & 0 \end{bmatrix}$$

A permutation is required here and we reorder the first, second and third rows.

The permutation matrix P is given by:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying this permutation matrix to A gives:

$$PA = \begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 2 & 1 \\ 4 & 2 & 6 & 0 \end{bmatrix}$$

We can now proceed with the LU decomposition. Subtract 2 times the first row from the fourth row by elimination matrix e:

$$e(PA) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} (PA) = \begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

This is our upper triangular matrix U. The lower triangular matrix L is given by the inverse of e:

$$U := \begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad L := e^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

We have a linear system of equation as $AX = B$, and then $PAX = PB$. But the PLU decomposition of A is given by $PA = LU$. The solution process now proceeds in two stages.

First Stage: *Solving $LY = PB$ for $Y = UX$.*

Second Stage: *Solving $UX = Y$.*

For the given system,

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ -4 \\ 1 \\ -4 \end{bmatrix}.$$

Solving $LY = PB$ yields:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} Y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -4 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 4 \\ -4 \end{bmatrix} \quad \Rightarrow \quad Y = \begin{bmatrix} -4 \\ 1 \\ 4 \\ 4 \end{bmatrix}$$

Solving $UX = Y$ gives the solution to the original system of equations:

$$\begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} X = \begin{bmatrix} -4 \\ 1 \\ 4 \\ 4 \end{bmatrix} \quad \Rightarrow \quad X = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

Therefore, the solution to the original system of equations is $x = -2$, $y = -1$, $z = 1$ and $w = 2$.

Example 3.3 *Consider a 3×4 matrix*

$$A = \begin{bmatrix} 1 & 2 & -3 & 1 \\ 2 & 4 & 0 & 7 \\ -1 & 3 & 2 & 0 \end{bmatrix},$$

for which pure LU-factorization does not exist. Multiplying A by the product of two matrices

$$E_{31}(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad E_{21}(-2) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we obtain

$$E_{21}(-2)E_{31}(1)A = \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 0 & 6 & 5 \\ 0 & 5 & -1 & 1 \end{bmatrix}.$$

To interchange the last two rows, we multiply by the matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

yielding

$$PE_{21}(-2)E_{31}(1)A = \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 5 & -1 & 1 \\ 0 & 0 & 6 & 5 \end{bmatrix} = U.$$

From this equation, we find the inverse matrix

$$(PE_{21}(-2)E_{31}(1))^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix},$$

which is not a triangular matrix. Therefore, we get the PLU decomposition:

$$A = PLU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 5 & -1 & 1 \\ 0 & 0 & 6 & 5 \end{bmatrix}.$$

Note that $P^2 = I$, the identity matrix, and $PA = LU$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 5 & -1 & 1 \\ 0 & 0 & 6 & 5 \end{bmatrix}.$$

When working with Gaussian elimination to perform LU decomposition, you might encounter a zero pivot element, which can cause the method to fail. However, if the matrix is nonsingular, there must be a nonzero element somewhere in that column, and you can use a permutation matrix to reorder the rows and move that nonzero element into the pivot position.

This idea is formalized in the LU decomposition with partial pivoting, where a permutation matrix P is used to reorder the rows of A to ensure that the decomposition can proceed without running into zero pivot elements.

4 Lower-Diagonal-Upper (LDU) Decomposition

LDU decomposition, also known as Lower-Diagonal-Upper decomposition, is a matrix factorization method expressed as:

$$A = LDU,$$

where D is a diagonal matrix, and L is a Lower triangular matrix and U is an upper triangular matrix that the diagonal elements of L and U are all ones, while the remaining elements can be arbitrary.

Example 4.1 *For the following matrix A we will find both LDU factorisation.*

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 4 & 4 & 6 \\ -2 & 3 & 1 \end{bmatrix}$$

In example 2.1, *there is a LU factorization for the matrix A as follows:*

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & -2 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 \\ 4 & 4 & 6 \\ -2 & 3 & 1 \end{bmatrix}$$

To obtain U in LDU format, we can modify U by dividing each row of U by its corresponding diagonal element. This adjustment ensures that all entries on the diagonal of U become one. We have :

$$U = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & -2 \\ 0 & 0 & 9 \end{bmatrix}$$

Now, we can present matrix A as follows:

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & -2 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

That is an LDU factorization of A.

4.1 The LDU decomposition for a symmetric matrix

A symmetric matrix A, this decomposition can be represented as:

$$A = LDL^T$$

where L is a lower triangular matrix with unit diagonal, D is a diagonal matrix, and L^T is the transpose of L.

Example 4.2 *Consider the symmetric matrix:*

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

We can decompose A into the form LDL^T:

$$A = LDL^T = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

5 The Cost of Elimination

The elimination process can be analyzed in stages, each with its specific computational costs. We'll take a closer look at the method and its efficiency by breaking down the stages and the mathematical implications.

Stage 1: Zeroing the First Column

The first stage of elimination targets column 1, creating zeros below the first pivot. To compute each new entry below the pivot row, a multiplication and a subtraction are required. This stage is counted as n^2 multiplications and n^2 subtractions. However, it's slightly less, precisely $n^2 - n$, as row 1 remains unaltered.

Subsequent Stages: Progressing Through the Columns

The next stage proceeds to clear the second column below the second pivot, leaving a working matrix of size $n - 1$. This stage can be estimated as $(n - 1)^2$ multiplications and subtractions.

As elimination continues, the matrices shrink. The rough count to reach V becomes the sum of squares $n^2 + (n-1)^2 + \ldots + 2^2 + 1^2$.

An exact formula for this sum of squares is $\frac{n(n+1)(2n+1)}{6}$. For large n, the lower-order terms are not critical, and the dominating number is $\frac{n^3}{3}$. This sum of squares bears resemblance to the integral of x^2, which from 0 to n equals $\frac{n^3}{3}$.

Handling the Right Side h

In the forward process, we subtract multiples of b_1 from the lower components $b_2, \ldots, b_n$ over $n-1$ steps. The second stage takes $n-2$ steps, excluding b_1, and the final stage of forward elimination takes a single step.

Back Substitution

The back substitution starts with computing X_n in one step (dividing by the last pivot), followed by two steps for the next unknown. By the time we reach X_1, it will need n steps ($n-1$ substitutions for the other unknowns, then a division by the first pivot). The exact count on the right side, progressing from h to c to x – down and back up – sums to n^2:

$$(n-1) + (n-2) + \ldots + 1 + [1 + 2 + \ldots + (n-1) + n] = n^2.$$

To understand this sum, pair $(n-1)$ with 1 and $(n-2)$ with 2, etc. This leaves n terms, each equal to n, making a total of n^2. The costs on the right side are notably lower than those on the left side, illustrating the computational efficiency of the method.