Matrices

Problem Sheet, Monday 31.07.2023 (Room U358)

Elementary Matrices

An elementary matrix is a unique category of square matrix which, upon left multiplication to a given matrix, administers one of these elementary row transformations:

1. Swapping two specific rows,
2. Multiplying a single row by a non-zero scalar,
3. Adding a scaled version of one row to another row.

These operations are pivotal in linear algebra, playing critical roles in the calculation of a matrix's inverse or determinant, and in solving systems of linear equations.

Below are some exercises designed to enhance your understanding and application of elementary matrices:

Question 1: Row Exchange

Design an elementary matrix E that, upon matrix multiplication to a 4×4 matrix, interchanges the 2nd and 4th rows. Implement this operation and subsequently find the reverse of E by reversing this transformation. Examine the results and derive your conclusions.

$$EA = E\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}.$$

Question 2: Row Scaling by a Nonzero Scalar

Design an elementary matrix E that, upon matrix multiplication to a 4×4 matrix, scales the 3rd row by a factor of 5. Execute this operation and subsequently find the reverse of E. Analyse the results and infer your conclusions.

$$EA = E\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 5a_{31} & 5a_{32} & 5a_{33} & 5a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Question 3: Addition of a Scaled Version of One Row to Another Row

Design an elementary matrix E that, upon matrix multiplication to a 4×4 matrix, adds three times the 1st row to the 2nd row. Perform this operation and subsequently reverse it to find E^{-1}. Review the outcomes and formulate your conclusions.

$$EA = E\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 3a_{11}+a_{21} & 3a_{12}+a_{22} & 3a_{13}+a_{23} & 3a_{14}+a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Question 4:

Based on the elementary matrices we obtained from the previous question, let's consider multiplying each of these elementary matrices with a 4×4 matrix from the right AE. What transformations do you anticipate these matrices will induce? Please explain your thoughts and provide examples or reasoning to support your answer."