Determinant

Problem Sheet, Friday 14.08.2023 (Room U358)

Question 1: An *orthogonal matrix* is a square matrix with real entries such that its transpose is also its inverse. Formally, for an orthogonal matrix A, we have:

$$A^T A = AA^T = I$$

where A^T denotes the transpose of A and I is the identity matrix of the same order as A. Thus the inverse of an orthogonal matrix is its transpose.

The columns (and rows) of an orthogonal matrix form an orthonormal set, implying that they are mutually orthogonal and each has unit length.

1. **Preservation of Dot Product**: Prove that if A is orthogonal, then for any vectors x and y in $\mathbb{R}^n$,

   $$x.y = (Ax).(Ay)$$

   This property indicates that the orthogonal transformation preserves the angles between vectors.

2. **Determinant Value**: Prove that the determinant of an orthogonal matrix is either $+1$ or -1, and specify what each value corresponds to in geometric terms.

3. **Preservation of Lengths**: Prove that for any vector x in $\mathbb{R}^n$ and an orthogonal matrix A, the lengths before and after transformation are preserved: $||Ax|| = ||x||$.

4. **Orthogonal Matrix Form an Orthonormal Set**: Prove that the columns (and by symmetry, rows) of an orthogonal matrix form an orthonormal set. Specifically, demonstrate the following:

   (a) They are mutually orthogonal.

   (b) Each has unit length.

Question 2: Prove that a transformation represented by a 2×2 matrix A is area-preserving if and only if $|\det(A)| = 1$.

Given the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

determine whether the transformation it represents is area-preserving.

Question 3: Suppose A is an $n \times n$ matrix and B is the matrix derived from A by interchanging any two rows (or columns). Prove that $\det(B) = -\det(A)$.