Linear Transformation

Problem Sheet, Friday 11.08.2023 (Room U358)

Question 1: A projection operator P maps a vector onto a subspace, creating its "shadow" or the closest point in that subspace. This is represented as:

$$P(\mathbf{v}) = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$$

Where $\mathbf{v}$ is the vector being projected, and $\mathbf{u}$ is the vector spanning the subspace. Verify whether the Projection Operator is a linear transformation, and explain what makes it so.

Question 2: Consider the integral operator $T : C([a,b]) \to \mathbb{R}$, where $C([a,b])$ is the set of continuous functions on the interval $[a,b]$, defined by

$$T(f) = \int_a^b f(x)\,dx$$

for each $f \in C([a,b])$. Determine whether T is a linear transformation.

Question 3: Verify whether the transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} 2x-y \\ y+z \end{bmatrix}$ is a linear transformation.

Question 4: Consider a basis in $\mathbb{R}^3$, denoted as $B = \{b_1, b_2, b_3\}$, where

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad b_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

A vector $\mathbf{v}$ in $\mathbb{R}^3$ is given in the standard basis as

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

What are the coordinates of this vector in the new basis B?

Question 5: Let's consider two sets of basis vectors in three-dimensional space ($\mathbb{R}^3$):

Basis Q is represented by vectors:

$$q_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad q_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Basis R is represented by vectors:

$$r_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad r_3 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

Also, consider the vector $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$.

You are asked to:

1. Compute the change-of-basis matrix that transitions from basis Q to basis R.
2. Represent the vector $\mathbf{v}$ in the R basis.